Review - Solutions of Ordinary Differential Equations

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First-Order, Separable, Ordinary Differential Equations

Example: $y = y(t)$	$y\frac{dy}{dt} = C$	• ***Separate variables***: $ydy = Cdt$ • Solve: $y^2/2 = Ct + C_1$
Example: $y = y(t)$	$ty\frac{dy}{dt} = C$	• ***Separate variables***: $ydy = \frac{C}{t}dt$ • Solve: $y^2/2 = C\ln(t) + C_1$

First-Order, Linear, Non-Homogeneous, Ordinary Differential Equations

Example: $y = y(x)$	$\frac{dy}{dx} + P(x)y = Q(x)$	• Multiply each term by an ***integrating factor***, $v = \exp\left(\int Pdx\right)$ • Solve: This leads to an exact differential on the LHS, which can be solved easily.

Higher-Order, Linear, Homogeneous, Ordinary Differential Equations with Constant Coefficients

Example: $y = y(x)$	$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$	• Rewrite in terms of a ***linear operator***, $D^n y \equiv \frac{d^n y}{dx^n}$: Here, $(D^3 - 4D^2 + D + 6)\,y = 0$ • Factor the LHS: Here, $(D + 1)(D - 2)(D - 3)\,y = 0$ • Find the roots of the LHS: Here the roots are -1, 2, and 3 • Solve: $y = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{3x}$
Example: $y = y(x)$	$\frac{d^2y}{dx^2} - a^2 y = 0$	• Rewrite in terms of the linear operator: Here, $(D^2 - a)\,y = 0$ • Factor the LHS: Here, $(D - a)(D + a)\,y = 0$ • Find the roots of the LHS: Here the roots are a and $-a$ • Solve: $y = C_1 e^{ax} + C_2 e^{-ax}$ or $y = C_3\cosh(ax) + C_4\sinh(ax)$
Example: $y = y(x)$	$\frac{d^2y}{dx^2} + a^2 y = 0$	• Rewrite in terms of the linear operator: Here, $(D^2 + a)\,y = 0$ • Factor the LHS: Here, $(D - ia)(D + ia)\,y = 0$ • Find the roots of the LHS: Here the roots are ia and $-ia$ • Solve: $y = C_1 e^{iax} + C_2 e^{-iax}$ or $y = C_3\cos(ax) + C_4\sin(ax)$

Higher-Order, Nonlinear, Non-Homogeneous, Ordinary Differential Equations

Example: $y = y(x)$	$\frac{d^3y}{dx^3} + P(x)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = Q(x)$	• Define N unknowns, where N is the highest order derivative of the original equation: Here, $N = 3$, and define $F_1 \equiv \frac{d^2y}{dx^2}$, $F_2 \equiv \frac{dy}{dx}$, $F_3 \equiv y$ • Split equation into N first-order equations, one for each F: $\frac{dF_1}{dx} = \frac{d^3y}{dx^3} = Q(x) - P(x)\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2$, $\frac{dF_2}{dx} = \frac{d^2y}{dx^2}$, $\frac{dF_3}{dx} = \frac{dy}{dx}$ • Define derivative array in standard Runge-Kutta form, in terms of the N unknowns, F_1, F_2, ..., F_N: $D_1 \equiv \frac{dF_1}{dx} = Q(x) - P(x)F_1 - F_2^2$, $D_2 \equiv \frac{dF_2}{dx} = F_1$, $D_3 \equiv \frac{dF_3}{dx} = F_2$ • Solve simultaneously for F_1, F_2, ..., F_N using the Runge-Kutta (R-K) numerical technique and the boundary conditions. [See a separate on-line learning module from Professor Cimbala that explains the R-K technique.]