Equation Sheet for ME 320

Print out for homework, quizzes, exams, and future reference. Author: John M. Cimbala, Penn State University. Latest revision, 06 December 2022

Notation for this equation sheet: V = volume, V = velocity, v = y-component of velocity, ν = kinematic viscosity

General and conversions: $g = 9.807\frac{\text{m}}{\text{s}^2}$, $\frac{2\pi\ \text{rad}}{\text{rotation}}$, $\frac{0.3048\ \text{m}}{1\ \text{ft}}$, $\frac{1\ \text{mile}}{1609.3\ \text{m}}$, $\frac{1\ \text{m}^3}{1000\ \text{L}}$, $\frac{\text{kg}\cdot\text{m}}{\text{s}^2\cdot\text{N}}$, $\frac{1\ \text{Pa}\cdot\text{m}^2}{1\ \text{N}}$, $\frac{1\ \text{kPa}\cdot\text{m}^2}{1\ \text{kN}}$, $\frac{1\ \text{kN}\cdot\text{m}}{1\ \text{kJ}}$, $\frac{1\ \text{kW}\cdot\text{s}}{1\ \text{kJ}}$, $\frac{1\ \text{Btu}}{1.055056\ \text{kJ}}$, $\frac{1\ \text{kg}}{2.205\ \text{lbm}}$, $\frac{1\ \text{ton}}{2000\ \text{lbm}}$, $\frac{1\ \text{tonne (metric ton)}}{1000\ \text{kg}}$, $\frac{1\ \text{g}}{10^6\ \mu\text{g}}$, $\frac{1\ \text{m}}{10^6\ \mu\text{m}}$, $T(\text{K}) \cong T(^\circ\text{C}) + 273.15$, $\mathsf{V}_{\text{sphere}} = \frac{4}{3}\pi(R_p)^3 = \frac{1}{6}\pi(D_p)^3$.

Molecular weights and moles: $m = nM$, $M_{\text{air}} = 28.97\ \text{g/mol}$, $M_{\text{water}} = 18.02\ \text{g/mol}$, Avagadro's number: 6.02214×10^{23}.

Standard ambient temperature and pressure (SATP): $P_{\text{SATP}} = 101.325\ \text{kPa} = 760\ \text{mm Hg} \quad T_{\text{SATP}} = 25^\circ\text{C} = 298.15\ \text{K}$.

Air at SATP: $\rho = 1.184\ \text{kg/m}^3$, $\lambda = 0.06704\ \mu\text{m}$, $\mu = 1.849\times10^{-5}\ \text{kg/(m s)}$.

Water at SATP: $\rho_{\text{water}} = 997.0\ \text{kg/m}^3,\ \mu_{\text{water}} = 0.891\times10^{-3}\ \text{kg/(m s)}$. Mercury at SATP: $\rho_{\text{mercury}} = 13534\ \text{kg/m}^3$.

Air at any *T*: $P = \rho R_{\text{air}} T$, **Sutherland**: $\mu \approx \mu_s \left(\frac{T}{T_{s,0}}\right)^{3/2} \frac{T_{s,0}+T_s}{T+T_s}$ where $T_{s,0}$ = 298.15 K, T_s = 110.4 K, $\mu_s = 1.849\times10^{-5}\ \frac{\text{kg}}{\text{m}\cdot\text{s}}$.

Ideal gas: $P\mathsf{V} = nR_uT = mRT$, $R = \frac{R_u}{M}$, $P = \rho RT$, $R_u = 8.314\frac{\text{kJ}}{\text{kmol}\cdot\text{K}}$, $R_{\text{air}} = 0.2870\frac{\text{kJ}}{\text{kg}\cdot\text{K}} = 287.0\frac{\text{m}^2}{\text{s}^2\cdot\text{K}}$, $u = c_v T$, $h = c_P T$, $c_P - c_v = R$, $k = \frac{c_P}{c_v}$, $c_v = \frac{R}{k-1}$, $c_P = \frac{Rk}{k-1}$, $s_2 - s_1 = c_P \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1} = c_v \ln\frac{T_2}{T_1} + R\ln\frac{\upsilon_2}{\upsilon_1}$.

Air: $k = 1.40$, $c_v = 717.5\frac{\text{J}}{\text{kg}\cdot\text{K}} = 0.7175\frac{\text{kJ}}{\text{kg}\cdot\text{K}} = 717.5\frac{\text{m}^2}{\text{s}^2\cdot\text{K}}$, $c_P = 1004.5\frac{\text{J}}{\text{kg}\cdot\text{K}} = 1.0045\frac{\text{kJ}}{\text{kg}\cdot\text{K}} = 1004.5\frac{\text{m}^2}{\text{s}^2\cdot\text{K}}$.

Thermodynamics of gases: $e = u + \frac{V^2}{2} + \cancel{gz}$, $\upsilon = \frac{1}{\rho}$, $h = u + P\upsilon = u + \frac{P}{\rho}$, $c_v = \left(\frac{\partial u}{\partial T}\right)_v$, $c_P = \left(\frac{\partial h}{\partial T}\right)_P$, $Tds = du + Pd\mathsf{V}$, $Tds = dh - \mathsf{V}dP$,

speed of sound = $c = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s}$ for any fluid. For an ***ideal gas***, $c = \sqrt{kRT}$, $k = \frac{c_P}{c_v}$, Mach number = $\text{Ma} = \frac{V}{c}$.

Density: $\rho = \frac{m}{\mathsf{V}}$. **Specific gravity**: $SG = \frac{\rho}{\rho_{\text{ref}}}$ where $\rho_{\text{ref}} = \rho_{H_2O} = 1000\frac{\text{kg}}{\text{m}^3}$ or $\rho_{\text{standard dry air}} = 1.29\frac{\text{kg}}{\text{m}^3}$.

Properties related to density: Specific weight $\gamma_s = \frac{W}{\mathsf{V}} = \rho g$; compressibility $\Delta\rho \approx \rho\left(\frac{1}{\kappa}\Delta P - \beta\Delta T\right)$ where β = volume expansion coefficient $\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_P \approx -\frac{\Delta\rho}{\rho\Delta T}$ where $\beta = \frac{1}{T}$ for an ideal gas, and κ = coefficient of compressibility $\kappa = \frac{1}{\alpha} = \rho\left(\frac{\partial P}{\partial\rho}\right)_T \approx \frac{\rho\Delta P}{\Delta\rho}$ where $\kappa = P$ for an ideal gas.

Simple shear flow and viscosity: For flow sandwiched between two infinite flat plates, $u(y) = \frac{Vy}{h}$. For $u = u(y)$, $\tau = \mu\frac{du}{dy}$.

Surface tension: $\Delta P_{\text{droplet}} = P_{\text{inside}} - P_{\text{outside}} = 2\frac{\sigma_s}{R}$, $\Delta P_{\text{bubble}} = P_{\text{inside}} - P_{\text{outside}} = 4\frac{\sigma_s}{R}$. **Capillary tube**: $h = \frac{2\sigma_s}{\rho g R}\cos\phi$.

Gage, vacuum, and vapor pressure: $P_{\text{gage}} = P_{\text{absolute}} - P_{\text{atm}}$, $P_{\text{vacuum}} = P_{\text{atm}} - P_{\text{absolute}}$, $P_v = 2.339\ \text{kPa for water at}\ T = 20^\circ\text{C}$.

Hydrostatics: $\vec{\nabla}P = \rho\vec{g}$, $\frac{dP}{dz} = -\rho g$, $P_{\text{below}} = P_{\text{above}} + \rho g|\Delta z|$, $\Delta P = \rho g h$. Useful applications: **Hydraulic jack**: $\frac{F_2}{F_1} = \frac{A_2}{A_1}$, **barometer**: $P_{\text{atm}} = \rho_{\text{Hg}} g h$, **manometer**: use $P_{\text{below}} = P_{\text{above}} + \rho g|\Delta z|$ successively all the way around the manometer.

Forces on submerged, plane surfaces: Using gage pressures, $F = P_C A = P_{\text{avg}} A$ where $y_{CP} = y_C + \frac{I_{xx,C}}{y_C A}$.

Forces on submerged, curved plates: Using gage pressures and projected areas, $F_H = F_x$, $F_V = F_y \pm W$, $F_R = \sqrt{F_H^{\,2} + F_V^{\,2}}$.

Buoyancy: (with "*f*" for fluid) **buoyant force**: $F_B = \rho_f g V_{\text{sub}}$, **weight**: $W = \rho_{\text{body}} g V_{\text{total}}$, **partially submerged**: $\frac{V_{\text{sub}}}{V_{\text{total}}} = \frac{\rho_{\text{body}}}{\rho_f}$.

Rigid body acceleration: $\vec{\nabla}P = \rho\vec{G} = \rho(\vec{g} - \vec{a})$. Use modified gravity vector $\vec{G}$ in place of $\vec{g}$ everywhere.

Material derivative: General case: $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{V}\cdot\vec{\nabla})$; acceleration: $\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}$.

Fluid flow patterns: Streamline for two-dimensional flow: $\left.\frac{dy}{dx}\right)_{\text{along streamline}} = \frac{v}{u}$; timeline: $x \cong u\Delta t$.

Kinematics: **Rate of translation**: $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$, **angular velocity**: $\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$,

vorticity: $\vec{\zeta} = \vec{\nabla}\times\vec{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z$,

linear strain rate: $\varepsilon_{xx} = \frac{\partial u}{\partial x},\quad \varepsilon_{yy} = \frac{\partial v}{\partial y},\quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$, **volumetric strain rate**: $\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$,

strain rate tensor:
$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}.$$

Reynolds transport theorem: $\frac{dB_{\text{system}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b\, dV + \int_{\text{CS}} \rho b \vec{V}_r \cdot \vec{n}\, dA$ where $\vec{V}_r = \vec{V} - \vec{V}_{\text{CS}}$.

Volume and mass flow rate: *Note*: Q and $\dot{V}$ are interchangeable $\dot{m}_{A_c} = \int_{A_c} \rho(\vec{V}\cdot\vec{n})\, dA_c$, $\dot{V}_{A_c} = \int_{A_c} (\vec{V}\cdot\vec{n})\, dA_c$, $Q = \dot{V} = VA_c$, $\dot{m} = \rho Q = \rho\dot{V}$.

Conservation of mass: $\frac{d}{dt} m_{\text{CV}} = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m}$. For steady flow, $\sum_{\text{in}} \dot{m} = \sum_{\text{out}} \dot{m}$. For incompressible flow, $\sum_{\text{in}} \dot{V} = \sum_{\text{out}} \dot{V}$.

Conservation of energy: SSSF form: $\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \sum_{\text{out}} \dot{m}\left(h + \alpha\frac{V^2}{2} + gz\right) - \sum_{\text{in}} \dot{m}\left(h + \alpha\frac{V^2}{2} + gz\right)$ (α is the kinetic energy correction factor).

Head form of energy equation: $\frac{P_1}{\rho_1 g} + \alpha_1\frac{V_1^2}{2g} + z_1 + h_{\text{pump, }u} = \frac{P_2}{\rho_2 g} + \alpha_2\frac{V_2^2}{2g} + z_2 + h_{\text{turbine, }e} + h_L$, where 1 = inlet, 2 = outlet, and the useful ("*u*") pump head and extracted ("*e*") turbine head are $h_{\text{pump, }u} = \frac{\eta_{\text{pump}}\dot{W}_{\text{pump shaft}}}{\dot{m}g}$ and $h_{\text{turbine, }e} = \frac{\dot{W}_{\text{turbine shaft}}}{\eta_{\text{turbine}}\dot{m}g}$.

Grade lines: ***Energy Grade Line*** = $\text{EGL} = \frac{P}{\rho g} + \frac{V^2}{2g} + z$, ***Hydraulic Grade Line*** = $\text{HGL} = \frac{P}{\rho g} + z$.

Beloved Bernoulli equation: $\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$ or $\frac{P}{\rho g} + \frac{V^2}{2g} + z = \text{constant along a streamline}$.

Momentum equation, fixed CV: $\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$

(β is the momentum flux correction factor). For a ***moving CV***, replace $\vec{V}$ with $\vec{V}_r = \vec{V} - \vec{V}_{CS}$ in the last two terms above.

Angular momentum equation, fixed CV: $\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V}) \rho dV + \sum_{\text{out}} (\vec{r} \times \dot{m} \vec{V}) - \sum_{\text{in}} (\vec{r} \times \dot{m} \vec{V})$. For steady case, the first term on the right side is zero and $\sum \vec{M} = \sum_{\text{out}} (\vec{r} \times \dot{m} \vec{V}) - \sum_{\text{in}} (\vec{r} \times \dot{m} \vec{V})$.

Dimensional analysis: A **nondimensional parameter** is called a Π. **Dynamic similarity** between model (m) and prototype (p): For $\Pi_1 = \text{fnc}(\Pi_2, \Pi_3, \ldots, \Pi_k)$, $\text{if } \Pi_{2,m} = \Pi_{2,p}, \Pi_{3,m} = \Pi_{3,p}, \ldots, \Pi_{k,m} = \Pi_{k,p}, \text{ then } \Pi_{1,m} = \Pi_{1,p}$. Method of repeating variables and Buckingham Pi theorem: For n = number of variables, k = number of Πs, and j = reduction, $k = n - j$.

Pipe flows: $\text{Re} = \frac{\rho V D}{\mu} = \frac{VD}{\nu}$, **hydraulic diameter**: $D_h = \frac{4A_c}{p}$, where A_c = cross-sectional area of the duct, p = wetted perimeter. **Mass flow rate**: $\dot{m} = \rho V A$. **Entrance length**: $\text{laminar: } \frac{L_h}{D} \approx 0.050\,\text{Re} \quad \text{turbulent: } \frac{L_h}{D} \approx 1.359\,\text{Re}^{1/4}$.

Darcy friction factor: $f = \frac{8\tau_w}{\rho V^2} = \text{fnc}\left(\text{Re}, \frac{\varepsilon}{D}\right)$. **Wall shear stress**: $\tau_{\text{wall}} = \mu \left.\frac{du}{dy}\right)_{\text{wall}}$. **Irreversible head losses**: $h_{L,\text{ total}} = \sum h_{L,\text{ major}} + \sum h_{L,\text{ minor}}$, **major head loss**: $h_{L,\text{ major}} = f \frac{L}{D}\frac{V^2}{2g}$, **minor head loss**: $h_{L,\text{ minor}} = K_L \frac{V^2}{2g}$.

- ***Fully developed laminar*** pipe flow, $\alpha = 2$, $f = \frac{64}{\text{Re}}$.
- ***Fully developed turbulent*** pipe flow, $\alpha \cong 1.05$, **Churchill equation**: $f = 8\left[(8/\text{Re})^{12} + (A+B)^{-1.5}\right]^{1/12}$, where parameters A and B are $A = \left\{2.457 \cdot \ln\left[\left(\frac{7}{\text{Re}}\right)^{0.9} + 0.27\frac{\varepsilon}{D}\right]\right\}^{16}$ and $B = \left(\frac{37530}{\text{Re}}\right)^{16}$ (or **Moody chart** with less accuracy).

Piping networks: **Series**: $\dot{V} = \text{constant} \qquad h_{L,\text{ total}} = \sum_i f_i \frac{L_i}{D_i}\frac{V_i^2}{2g} + \sum_j K_{L_j} \frac{V_j^2}{2g}$ (index i for each segment, j for each minor loss).

Parallel: conserve $\dot{V}$ at each junction; use separate energy equation for each branch. Solve all equations simultaneously.

Turbomachinery: ***Net head*** = $H = h_{\text{pump,u}} = \left(\frac{P}{\rho g} + \frac{V^2}{2g} + z\right)_{\text{out}} - \left(\frac{P}{\rho g} + \frac{V^2}{2g} + z\right)_{\text{in}}$, ***Brake horsepower*** = $\dot{W}_{\text{shaft}} = \text{bhp}$, ***Water horsepower*** = $\dot{W}_{\text{pump,u}} = \dot{m} g H = \rho \dot{V} g H$.

Pump and turbine performance parameters:

$C_Q = \text{Capacity coefficient} = \frac{\dot{V}}{\omega D^3}$, $C_H = \text{Head coefficient} = \frac{gH}{\omega^2 D^2}$, $C_P = \text{Power coefficient} = \frac{\text{bhp}}{\rho \omega^3 D^5}$,

$\eta_{\text{pump}} = \frac{\dot{W}_{\text{pump,u}}}{\text{bhp}} = \frac{\rho \dot{V} g H}{\text{bhp}} = \frac{C_Q C_H}{C_P} = \text{function of } C_Q$, $\eta_{\text{turbine}} = \frac{\text{bhp}}{\dot{W}_{\text{turbine,e}}} = \frac{\text{bhp}}{\rho \dot{V} g H} = \frac{C_P}{C_Q C_H} = \text{function of } C_P$.

Pump selection: To match a pump (***available head***) to a piping system (***required head***), match $H_{\text{available}} = H_{\text{required}}$.

Pump and turbine affinity laws: $\frac{\dot{V}_B}{\dot{V}_A}=\frac{\omega_B}{\omega_A}\left(\frac{D_B}{D_A}\right)^3$, $\frac{H_B}{H_A}=\left(\frac{\omega_B}{\omega_A}\right)^2\left(\frac{D_B}{D_A}\right)^2$, $\frac{\text{bhp}_B}{\text{bhp}_A}=\frac{\rho_B}{\rho_A}\left(\frac{\omega_B}{\omega_A}\right)^3\left(\frac{D_B}{D_A}\right)^5$.

Continuity equation: $\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\left(\rho\vec{V}\right)=0$. If ***incompressible***, $\vec{\nabla}\cdot\vec{V}=0$, which we expand for two coordinate systems:

Cartesian coordinates, $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$; **Cylindrical coordinates**, $\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}=0$.

2-D Stream function: Cartesian (x-y plane): $u=\frac{\partial\psi}{\partial y}$, $v=-\frac{\partial\psi}{\partial x}$; Cylindrical planar ($r$-$\theta$ plane): $u_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta}$, $u_\theta=-\frac{\partial\psi}{\partial r}$.

Navier-Stokes equation: (incompressible, Newtonian fluid) $\rho\frac{D\vec{V}}{Dt}=\rho\left[\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right]=-\vec{\nabla}P+\rho\vec{g}+\mu\nabla^2\vec{V}$.

Cartesian coordinates (x, y, z), (u, v, w):

x-component: $\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$,

y-component: $\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$,

z-component: $\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$. Also, $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$,

$\vec{V}\cdot\vec{\nabla}=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$, $\zeta_x=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}$, $\zeta_y=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}$, $\zeta_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$.

Cylindrical coordinates (r, θ, z), (u_r, u_θ, u_z):

r-component: $\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^{\,2}}{r}\right)=-\frac{\partial P}{\partial r}+\rho g_r+\mu\left(\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)$,

θ-component: $\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}\right)=-\frac{1}{r}\frac{\partial P}{\partial\theta}+\rho g_\theta+\mu\left(\nabla^2 u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right)$,

The θ-component of the Navier-Stokes equation with an alternate form of the viscous term:

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}\right)=-\frac{1}{r}\frac{\partial P}{\partial\theta}+\rho g_\theta+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{\partial^2 u_\theta}{\partial z^2}-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right),$$

z-component: $\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left(\nabla^2 u_z\right)$. Also, $\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$,

$\vec{V}\cdot\vec{\nabla}=u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}$, $\zeta_r=\frac{1}{r}\frac{\partial u_z}{\partial\theta}-\frac{\partial u_\theta}{\partial z}$, $\zeta_\theta=\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}$, $\zeta_z=\frac{1}{r}\frac{\partial}{\partial r}\left(ru_\theta\right)-\frac{1}{r}\frac{\partial u_r}{\partial\theta}$.

Incompressible Navier-Stokes equation in nondimensional form: (with L, f, V, P_o-P_∞ as appropriate scaling parameters)

$[\text{St}]\frac{\partial\vec{V}^*}{\partial t^*}+\left(\vec{V}^*\cdot\vec{\nabla}^*\right)\vec{V}^*=-[\text{Eu}]\vec{\nabla}^*P^*+\left[\frac{1}{\text{Fr}^2}\right]\vec{g}^*+\left[\frac{1}{\text{Re}}\right]\vec{\nabla}^{*2}\vec{V}^*$, where $\text{St}=\frac{fL}{V}$, $\text{Eu}=\frac{P_0-P_\infty}{\rho V^2}$, $\text{Fr}=\frac{V}{\sqrt{gL}}$, $\text{Re}=\frac{\rho VL}{\mu}$.

Creeping flow: (for Re << 1) $\vec{\nabla}P\approx\mu\nabla^2\vec{V}$, sphere drag: $F_D=3\pi\mu VL$, drag coefficient: $C_{D,\text{creeping}}=\frac{F_D}{\mu VL}$.

Cunningham correction factor: (for spheres) $\text{Kn}=\frac{\lambda}{D_p}$, $\lambda=\frac{\mu}{0.499}\sqrt{\frac{\pi}{8\rho P}}$, $C=1+\text{Kn}\left[2.514+0.80\exp\left(-\frac{0.55}{\text{Kn}}\right)\right]$.

Euler Eq. (inviscid regions of flow): $\rho\frac{D\vec{V}}{Dt}=\rho\left[\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right]=-\vec{\nabla}P+\rho\vec{g}$, $\frac{P}{\rho g}+\frac{V^2}{2g}+z=\text{constant along a streamline}$.

Potential (irrotational) flow: $\vec{\zeta} = \vec{\nabla} \times \vec{V} = 0$ → $\vec{V} = \vec{\nabla}\phi$ → $\nabla^2 \phi = 0$ & $\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant everywhere}$. ← "Most Beloved Bernoulli Equation"

***Cartesian* coordinates**: $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$. ***Cylindrical* coordinates**: $\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$.

If flow is also 2-D, then $\nabla^2\psi = 0$ as well. Superposition of both ϕ and ψ (and velocity vectors) is valid for potential flow.

2-D Potential flow: $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$, $\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$, $u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$, $v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$, $\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$.

In cylindrical coordinates, the above equations are $\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$, $\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$,

$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$, $u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$, $\zeta_z = \frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta} = 0$.

Boundary layers: $Re_x = \frac{\rho U x}{\mu} = \frac{Ux}{\nu}$, where x is *along* the body. $U(x)$ is the outer flow (just outside the boundary layer).

Steady flow continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, x-momentum: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$, $U\frac{dU}{dx} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$, y-momentum: $\frac{\partial P}{\partial y} \approx 0$.

Flat plate boundary layer:

If ***laminar*** ($\mathrm{Re}_x < 5\times10^5$), $\frac{\delta}{x} = \frac{4.91}{\sqrt{\mathrm{Re}_x}}$, $\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$, $\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$, $C_{f,x} = \frac{2\tau_w}{\rho U^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$, $C_f = C_D = \frac{1.33}{\sqrt{\mathrm{Re}_x}}$.

If ***turbulent*** and ***smooth*** ($\mathrm{Re}_x > 5\times10^5$), $\frac{\delta}{x} \approx \frac{0.38}{(\mathrm{Re}_x)^{1/5}}$, $\frac{\delta^*}{x} \approx \frac{0.048}{(\mathrm{Re}_x)^{1/5}}$, $\frac{\theta}{x} \approx \frac{0.037}{(\mathrm{Re}_x)^{1/5}}$, $C_{f,x} \approx \frac{0.059}{(\mathrm{Re}_x)^{1/5}}$, $C_f = C_D = \frac{0.074}{\mathrm{Re}_x^{1/5}}$.

Drag and lift on bodies: $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$, $C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A}$, where A = projected frontal area *or* planform area. C_D includes skin friction *and* pressure drag. **Bodies without ground effect**, $F_D = \frac{1}{2}\rho V^2 C_D A$ and required power = $\dot{W} = F_D V$.

Vehicles in ground effect, $F_{D,\text{total}} = \mu_{\text{rolling}} W + \frac{1}{2}\rho V^2 C_D A$ where μ_{rolling} = coefficient of rolling resistance, and W is the vehicle weight. The required ***power to the wheels*** = $\dot{W} = F_{D,\text{total}} V = \mu_{\text{rolling}} W V + \frac{1}{2}\rho V^3 C_D A$.

For a ***smooth sphere***: **Morrison**: $C_D \approx \frac{24}{\mathrm{Re}} + \frac{2.6\left(\frac{\mathrm{Re}}{5.0}\right)}{1+\left(\frac{\mathrm{Re}}{5.0}\right)^{1.52}} + \frac{0.411\left(\frac{\mathrm{Re}}{2.63\times10^5}\right)^{-7.94}}{1+\left(\frac{\mathrm{Re}}{2.63\times10^5}\right)^{-8.00}} + \frac{0.25\left(\frac{\mathrm{Re}}{10^6}\right)}{1+\left(\frac{\mathrm{Re}}{10^6}\right)}$ for $\mathrm{Re} < 10^6$, $\mathrm{Re} = \frac{\rho V_t D_p}{\mu}$

Terminal settling speed: $V_t = \sqrt{\frac{4}{3}\frac{\rho_p - \rho}{\rho} g D_p \frac{C}{C_D}}$, where iteration is required to solve for V_t since this equation is implicit.

Compressible flow:

Mach number Ma and speed of sound *c*: $\mathrm{Ma} = \frac{V}{c}$, $c = \sqrt{\left(\frac{\partial P}{\partial\rho}\right)_s}$, Ideal gas: $k = \frac{c_P}{c_v}$, $c = \sqrt{kRT}$.

(*a* sometimes instead of *c*; *M* sometimes instead of Ma; *γ* sometimes instead of *k*)

Adiabatic, isentropic, 1-D duct flow: $\frac{T_0}{T} = 1 + \frac{k-1}{2}\mathrm{Ma}^2$, $\frac{\rho_0}{\rho} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{1}{k-1}}$, $\frac{P_0}{P} = \left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{k}{k-1}}$, $\frac{a_0}{a} = \left(\frac{T_0}{T}\right)^{1/2}$,

$\frac{T_0}{T^*} = \frac{k+1}{2}$, $\frac{\rho_0}{\rho^*} = \left(\frac{T_0}{T^*}\right)^{\frac{1}{k-1}}$, $\frac{P_0}{P^*} = \left(\frac{T_0}{T^*}\right)^{\frac{k}{k-1}}$, $\frac{a_0}{a^*} = \left(\frac{T_0}{T^*}\right)^{1/2}$, $\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left[\left(\frac{2}{k+1}\right)\left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)\right]^{\frac{k+1}{2(k-1)}}$.

Mass flow rate: General case: $\dot{m} = P_0 A \mathrm{Ma}\sqrt{\frac{k}{RT_0}}\left(1+\frac{k-1}{2}\mathrm{Ma}^2\right)^{\frac{-(k+1)}{2(k-1)}}$, Choked case: $\dot{m} = \dot{m}_{\max} = P_0 A^*\sqrt{\frac{k}{RT_0}}\left(\frac{k+1}{2}\right)^{\frac{-(k+1)}{2(k-1)}}$.

Normal shock: (1 to 2) $T_{02} = T_{01}$, $\mathrm{Ma}_2 = \sqrt{\dfrac{(k-1)\mathrm{Ma}_1^2 + 2}{2k\mathrm{Ma}_1^2 - k + 1}}$, $\dfrac{P_2}{P_1} = \dfrac{1 + k\mathrm{Ma}_1^2}{1 + k\mathrm{Ma}_2^2} = \dfrac{2k\mathrm{Ma}_1^2 - k + 1}{k+1}$, $\dfrac{\rho_2}{\rho_1} = \dfrac{V_1}{V_2} = \dfrac{(k+1)\mathrm{Ma}_1^2}{2 + (k-1)\mathrm{Ma}_1^2}$,

$\dfrac{T_2}{T_1} = \dfrac{2 + (k-1)\mathrm{Ma}_1^2}{2 + (k-1)\mathrm{Ma}_2^2}$, $\dfrac{P_{0,2}}{P_{0,1}} = \dfrac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left[\dfrac{1 + (k-1)\mathrm{Ma}_2^2/2}{1 + (k-1)\mathrm{Ma}_1^2/2}\right]^{\frac{k+1}{2(k-1)}}$.

Oblique shock: (1 to 2) Use above shock equations except use *normal* components of Mach number, $\mathrm{Ma}_{1,n}$ and $\mathrm{Ma}_{2,n}$.

$\tan\theta = \dfrac{2\cot\beta\left(\mathrm{Ma}_1^2 \sin^2\beta - 1\right)}{\mathrm{Ma}_1^2\left[k + \cos(2\beta)\right] + 2}$, $\mathrm{Ma}_{1,n} = \mathrm{Ma}_1 \sin\beta$, $\mathrm{Ma}_{2,n} = \mathrm{Ma}_2 \sin(\beta - \theta)$, θ = turning angle, β = shock angle. Must iterate to solve for Ma_1.

Moody Chart: (Note: It is easier and more accurate to use the Churchill equation instead of reading off this chart, but this may be useful as a first guess in an iteration and/or for quick analyses.)

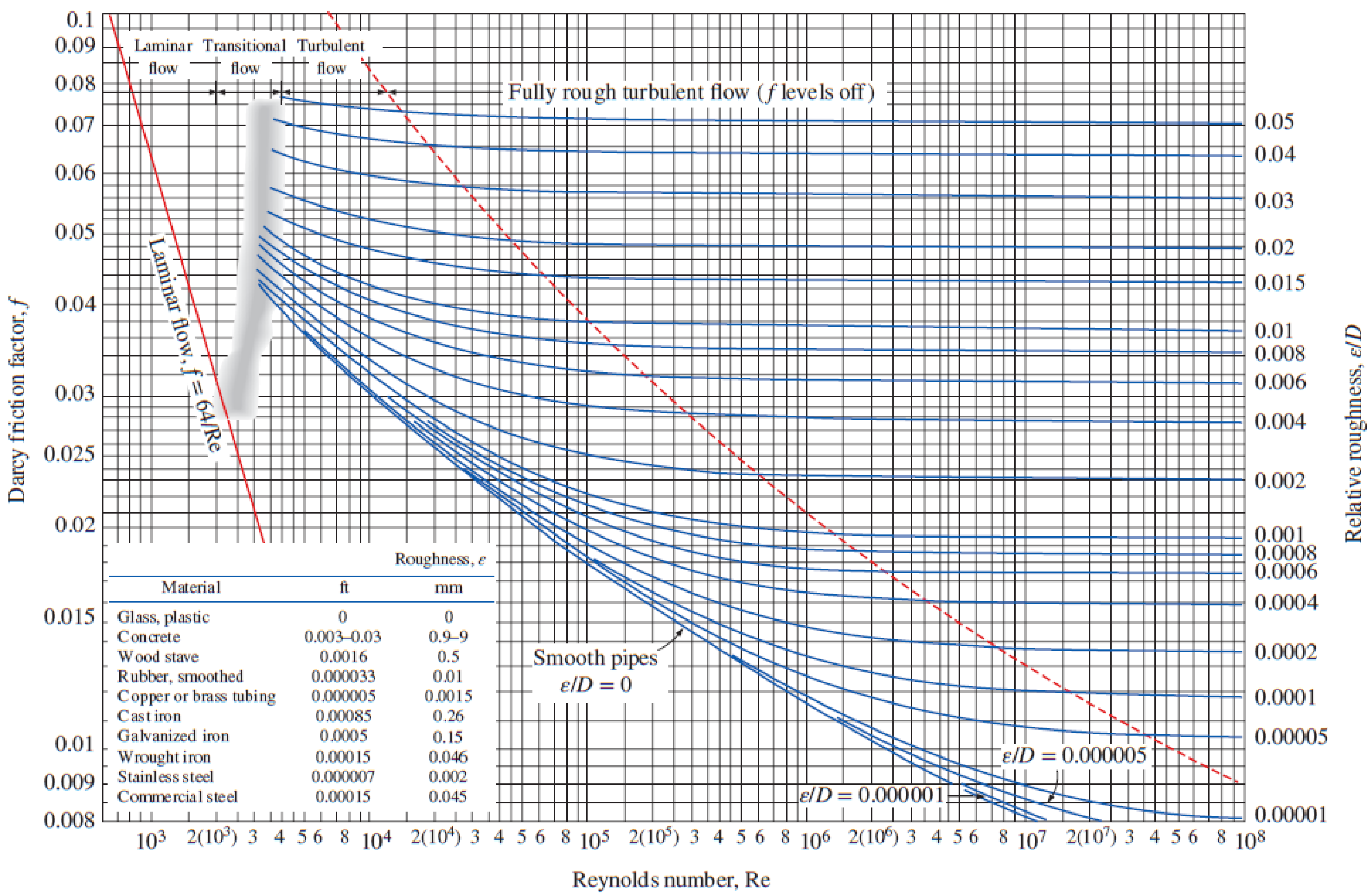

FIGURE A–12

The Moody chart for the friction factor for fully developed flow in circular pipes for use in the head loss relation $h_L = f\dfrac{L}{D}\dfrac{V^2}{2g}$. Friction factors in the turbulent flow are evaluated from the Colebrook equation $\dfrac{1}{\sqrt{f}} = -2\log_{10}\left(\dfrac{\varepsilon/D}{3.7} + \dfrac{2.51}{\mathrm{Re}\sqrt{f}}\right)$.