THE STRAIN RATE TENSOR

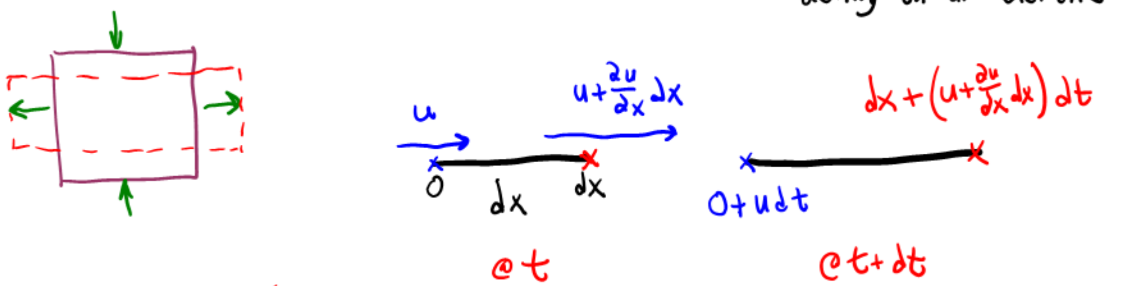
In this lesson, we will:

- Define **linear strain rate**, **volumetric strain rate**, and **shear strain rate**
- Discuss how to combine the linear and shear strain rates into the **strain rate tensor**
- Do some example problems

Linear Strain Rate

Linear strain is defined as the increase in length per unit length.

due to normal stresses acting on an element



$$\text{Linear strain} = \frac{\underbrace{\cancel{dx} + \cancel{u\,dt} + \frac{\partial u}{\partial x}\cancel{dx}\,dt - \cancel{u\,dt}}_{\text{new length}} \underbrace{- \cancel{dx}}_{\text{original length}}}{\cancel{dx}} = \frac{\partial u}{\partial x} dt$$

We are interested in the rate of linear strain $= \frac{d}{dt}\left(\frac{\partial u}{\partial x} dt\right) \approx \frac{1}{\cancel{dt}} \frac{\partial u}{\partial x} \cancel{dt}$

Thus, linear strain in x-direction $= \frac{\partial u}{\partial x}$

$$\boxed{\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}}$$

☆ LINEAR STRAIN RATES IN CARTESIAN COORDINATES

Example: Linear Strain Rate

Given: A steady, 2-D velocity field is given by components

$$u = 2xy$$
$$v = -y^2$$
$$w = 0$$

To do: Calculate the three linear strain rates.

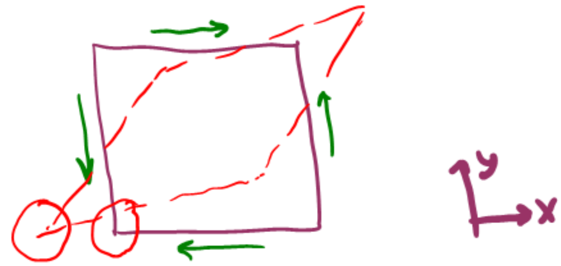
Solution: $\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$

$$\boxed{\varepsilon_{xx} = 2y \qquad \varepsilon_{yy} = -2y \qquad \varepsilon_{zz} = 0}$$

☆

Shear Strain Rate

Shear strain is defined as half of the decrease of the angle between two initially perpendicular lines that intersect at a point.



The rate of shear strain = shear strain rate

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

equal   equal   equal

$$\varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

6 components

but only 3 independent ones

Example: Shear Strain Rate

Given: A steady, 2-D velocity field is given by components

$$u = 2xy$$
$$v = -y^2$$
$$w = 0$$

To do: Calculate the six shear strain rates.

Solution: $\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$, $\quad \varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\left(\cancel{\frac{\partial w}{\partial x}} + \cancel{\frac{\partial u}{\partial z}}\right)$, $\quad \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\cancel{\frac{\partial v}{\partial z}} + \cancel{\frac{\partial w}{\partial y}}\right)$

$$= \frac{1}{2}(2x + 0)$$

$$\boxed{\varepsilon_{xy} = \varepsilon_{yx} = x} \qquad \boxed{\varepsilon_{zx} = \varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zy} = 0}$$

THERE IS SHEAR STRAIN IN THE X-Y PLANE ONLY ☆

The Strain Rate Tensor

2^{nd} order tensor (9-components); write as a 3x3 matrix

- Scalar → magnitude only (0th-order tensor)
- Vector → magnitude & direction (1st-order tensor)
- 2^{nd}-order tensor → magnitude & direction, depending on surface orientation

$$\vec{\vec{\varepsilon}} = \boldsymbol{\varepsilon} = \varepsilon_{ij} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

$1 \to x$
$2 \to y$
$3 \to z$

ε_{ij} will be important later when we derive the differential eq's of fluid motion

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

9 components
6 independent components

Example: Rates of motion and deformation **(Continuation of previous example)**

Given: A two-dimensional velocity field in the *x*-*y* plane: $\boxed{\vec{V} = (u, v) = 3x\vec{i} - 3y\vec{j}}$ $(w = 0)$.

To do: Calculate (**a**) rate of translation, (**b**) rate of rotation, (**c**) the three linear strain rates, (**d**) the six shear strain rates, and (**e**) the strain rate tensor.

Solution: We did Parts (**a**) and (**b**) in a previous lesson for this velocity field. Recall,

(**a**) The rate of translation is simply the velocity vector, $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$. Here, the rate of translation = $\boxed{\vec{V} = 3x\vec{i} - 3y\vec{j}}$.

(**b**) The rate of rotation is $\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$. Here, the rate of rotation is $\boxed{\vec{\omega} = 0}$, and the vorticity = $\vec{\zeta} = 2\vec{\omega} = 0$. This flow is *irrotational*.

(**c**) The three components of linear strain rate are

$u = 3x \quad v = -3y$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\boxed{\varepsilon_{xx} = 3 \qquad \varepsilon_{yy} = -3 \qquad \varepsilon_{zz} = 0}$$

☆ NOTICE: $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0$

WHEN THE SUM OF THE LINEAR STRAIN RATES = 0, THE FLOW IS INCOMPRESSIBLE ☆

(**d**) The three components of shear strain rate are

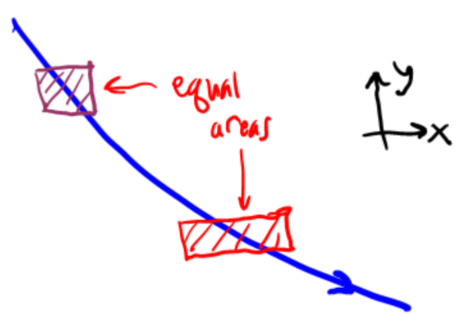
$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(0 + 0), \quad \varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0 \text{ (2-D)}, \quad \varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \text{ (2-D)}$$

$$\left[\begin{matrix} u = 3x \\ v = -3y \end{matrix}\right]$$

In this flow, there is no shear strain ☆

(**e**) The strain rate tensor

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \boxed{\begin{matrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{matrix}}$$



Here, the x-y axes are principal axes ☆

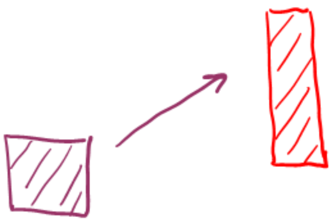
If we rotate axes, all of these components will change

Volumetric Strain Rate

Volumetric strain is defined as the change of volume of a fluid particle per unit volume following a fluid particle.

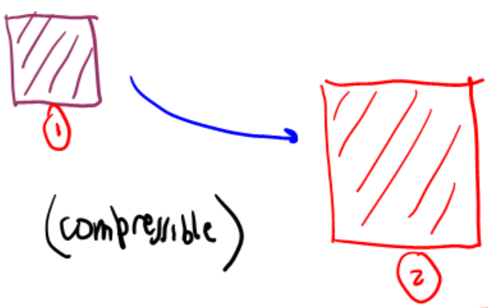
We are interested in the rate of this

Define $\boxed{\frac{1}{\forall}\frac{D\forall}{Dt} = \text{volumetric strain rate}}$



Equal volumes, $\therefore \boxed{\frac{1}{\forall}\frac{D\forall}{Dt} = 0}$

(incompressible)



Vol. 2 > Vol. 1

$\therefore \boxed{\frac{1}{\forall}\frac{D\forall}{Dt} > 0}$

(compressible)

If volumetric strain rate = 0 everywhere, the flow is incompressible

Example: ~~Linear~~ Volumetric Strain Rate

Given: A steady, 2-D velocity field is given by components

$$u = 2xy$$
$$v = -y^2$$
$$w = 0$$

To do: Use the volumetric strain rate to determine if this flow is compressible or incompressible.

Solution: $$\boxed{\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}$$

$$2y - 2y + 0 = 0$$

Since $\frac{1}{\forall}\frac{D\forall}{Dt} = 0$ everywhere

THIS FLOW IS INCOMPRESSIBLE