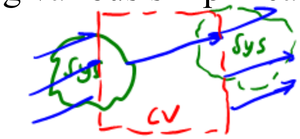
REYNOLDS TRANSPORT THEOREM

In this lesson, we will:

- Show an analogy between the material derivative and the **Reynolds Transport Theorem**
- Discuss the purpose and usefulness of the Reynolds Transport Theorem (**RTT**)
- Show alternate forms of the RTT, including various simplifications
- Do an example problem

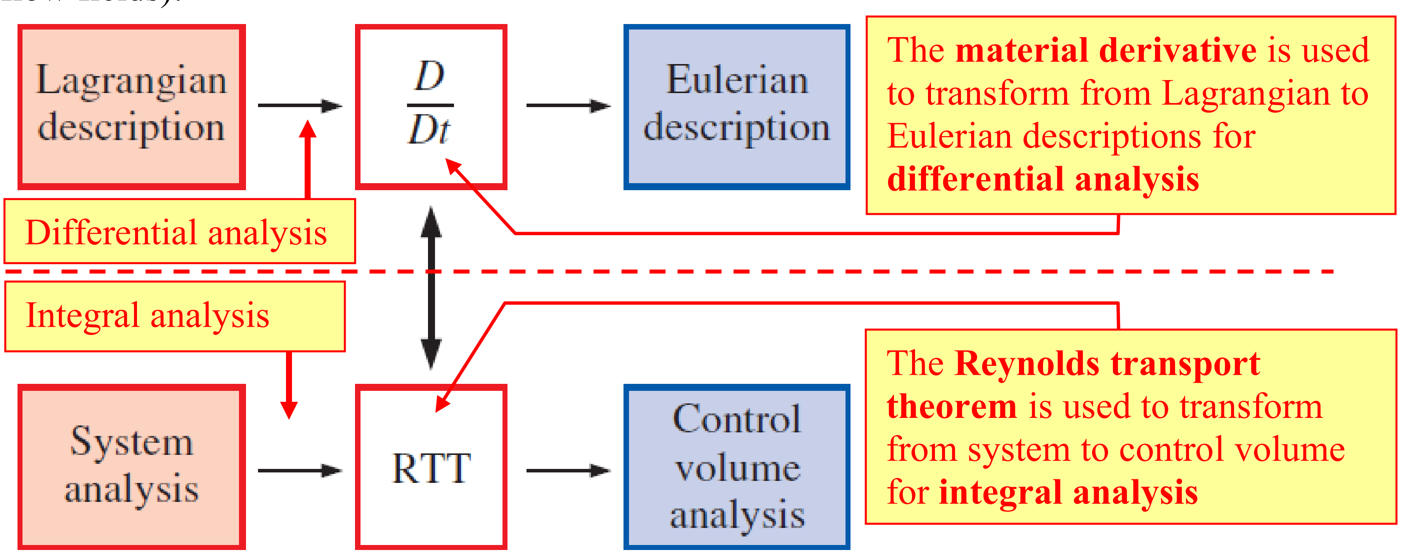


Introduction and Overview

- A ***system*** [also called a ***closed system***] is a quantity of matter of fixed identity. *No mass can cross a system boundary*.
- A ***control volume*** [also called an ***open system***] is a region in space chosen for study. *Mass **can** cross a control surface* (the surface of the control volume).
- The fundamental conservation laws (conservation of mass, energy, etc.) *apply directly to systems*.
- However, in most fluid mechanics problems, ***control volume analysis is preferred over system analysis*** (for the same reason that the Eulerian description is usually preferred over the Lagrangian description).
- Therefore, we need to *transform the conservation laws from a system to a control volume*. This is accomplished with the ***Reynolds transport theorem (RTT)***.

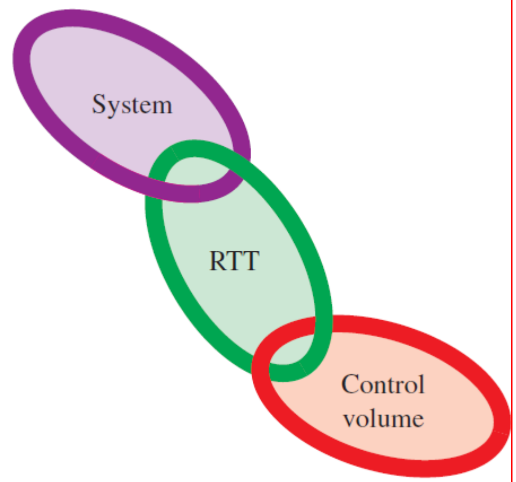
Analogy Between the Material Derivative and the RTT

There is a direct **analogy** between the transformation from **Lagrangian** to **Eulerian** descriptions (**for differential analysis** using infinitesimally small fluid elements) and the transformation from **systems** to **control volumes** (**for integral analysis** using large, finite flow fields):



In both cases, the fundamental laws of physics (conservation laws) are ***known*** and apply directly to the analysis on the left (Lagrangian or system).
These laws of physics must be ***transformed*** so as to be useful in the analysis on the right (Eulerian or control volume).

Another way to think about the RTT is that it is a ***link*** between the system approach and the control volume approach.



See textbook for detailed derivation of the RTT. Here are some highlights: B can be a scalar or a vector

- Let B represent any **extensive property** (like mass, energy, or momentum).
- Let b be the corresponding **intensive property**, i.e., $b = B/m$ (property B per unit mass).
- Our goal is to find a relationship between B_{sys} or b_{sys} (property of the system, for which we know the conservation laws) and B_{CV} or b_{CV} (property of the control volume, which we prefer to use in our analysis).
- The results are shown below in various forms:

change of B_{sys} due to unsteadiness

For **fixed** (non-moving and non-deforming) control volumes,

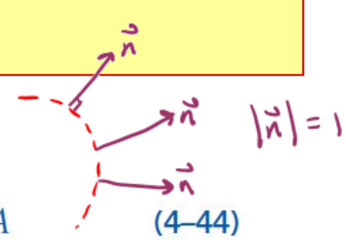
change of B_{sys} due to movement of fluid across the C.S.

RTT, fixed CV:
$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V}\cdot\vec{n} \, dA \qquad (4–41)$$

unit outward normal vector

Alternate RTT, fixed CV:
$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b) \, dV + \int_{CS} \rho b \vec{V}\cdot\vec{n} \, dA \qquad (4–42)$$

Since the control volume is *fixed*, the order of integration or differentiation does not matter, i.e., $\frac{d}{dt}\int_{CV} \ldots$ is the same as $\int_{CV}\frac{\partial}{\partial t} \ldots$. Thus, the two circled quantities above are *equivalent* for a fixed control volume.

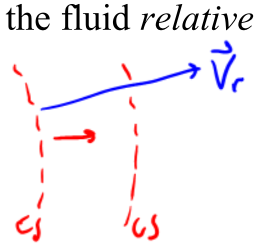


For **nonfixed** (moving and/or deforming) control volumes,

RTT, nonfixed CV:
$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho b \, dV + \int_{CS} \rho b \vec{V}_r\cdot\vec{n} \, dA \qquad (4–44)$$

Note: The only difference in the equations is that we replace $\vec{V}$ by $\vec{V}_r$ in this version of the RTT for a moving and/or deforming control volume.

where $\vec{V}_r$ is the **relative velocity**, i.e., the velocity of the fluid *relative to the control surface* (which may be moving or deforming),



Relative velocity:
$$\vec{V}_r = \vec{V} - \vec{V}_{CS} \qquad (4–43)$$

We can also switch the order of the time derivative and the integral in the first term on the right, but only if we use the *absolute* (rather than the relative) velocity in the second term on the right, i.e.,

Alternate RTT, nonfixed CV:
$$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n}\, dA \qquad \textbf{(4–45)}$$

Comparing Eqs. 4-45 and 4-42, we see that they are identical. Thus, the most general form of the RTT that *applies to both fixed and non-fixed control volumes* is

General RTT, nonfixed CV:
$$\frac{dB_{\text{sys}}}{dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\rho b)\, dV + \int_{\text{CS}} \rho b \vec{V} \cdot \vec{n}\, dA \qquad \textbf{(4–53)}$$

Even though this equation is most general, it is often easier *in practice* to use Eq. 4-44 for moving and/or deforming (non-fixed) control volumes because the algebra is easier.

Simplifications:

- For **steady** flow, the volume integral drops out. In terms of relative velocity,

  RTT, steady flow:
  $$\frac{dB_{\text{sys}}}{dt} = \oint_{\text{CS}} \rho b \vec{V}_r \cdot \vec{n}\, dA \qquad \textbf{(4–46)}$$

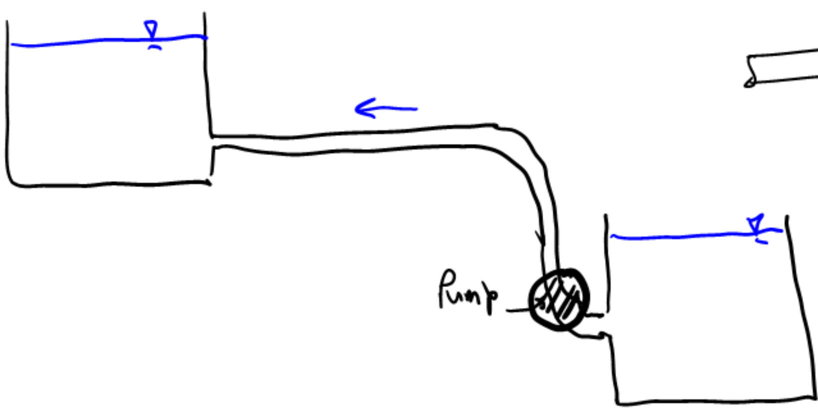
  indicates integration over the entire CS

- For control volumes where there are **well-defined inlets and outlets**, the control surface integral can be simplified, avoiding cumbersome integrations,

  Approximate RTT for well-defined inlets and outlets:

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b\, dV + \sum_{\text{out}} \underbrace{\rho_{\text{avg}} b_{\text{avg}} V_{r,\text{avg}} A}_{\text{for each outlet}} - \sum_{\text{in}} \underbrace{\rho_{\text{avg}} b_{\text{avg}} V_{r,\text{avg}} A}_{\text{for each inlet}} \qquad \textbf{(4–48)}$$

Note that the above equation is *approximate*, so it may not always be accurate; but it will be used almost exclusively in this course and is used generally in engineering analysis.



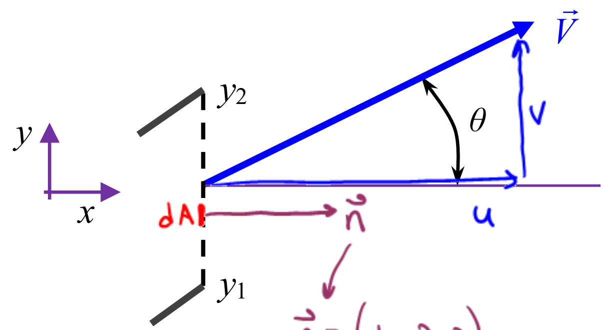
Sometimes we need to integrate over a portion of the CS.

Example: Reynolds Transport Theorem

Given: There is a steady, two-dimensional flow of water (density = 1000 kg/m^3) out of a large outlet as sketched (not to scale).

- Location y_1 = 1.00 m
- Location y_2 = 3.00 m
- Property b in the RTT equation is 1
- The width of the outlet (into the page in the sketch) is s = 10.0 m
- The velocity components are u = 3.22 m/s (horizontal direction) and v = 1.50 m/s (vertical direction)
- The velocity is constant across the entire outlet

$b = 1 \Rightarrow B = m$ (cons. of mass)



$\vec{n} = (1, 0, 0)$

$\vec{V} = (u, v, 0)$

To do: Calculate the RTT surface integral $\int_{\text{outlet}} \rho b\left(\vec{V}\cdot\vec{n}\right) dA$ across this outlet.

Solution:

$$\vec{V}\cdot\vec{n} = (u, v, 0)\cdot(1, 0, 0)$$

$$= u(1) + v\cancel{(0)} + 0\cancel{(0)}$$

$$\boxed{\vec{V}\cdot\vec{n} = u} \ \star$$

$$\int_{\text{outlet}} \rho b \underbrace{(\vec{V}\cdot\vec{n})}_{u} \underbrace{dA}_{s\,dy} = \rho b u s \int_{y_1}^{y_2} dy = \boxed{\rho b u s (y_2 - y_1)} \ \star$$

#s:

$\left(b = \frac{m}{m}\right)$ $(B = m)$

$$\int_{\text{outlet}} \rho b(\vec{V}\cdot\vec{n}) dA = \left(1000 \frac{\text{kg}}{\cancel{\text{m}^3}}\right)(1)\left(3.22 \frac{\cancel{\text{m}}}{\text{s}}\right)(10.0\ \cancel{\text{m}})(3.00 - 1.00)\ \cancel{\text{m}}$$

$$= \boxed{64{,}400\ \frac{\text{kg}}{\text{s}}}$$