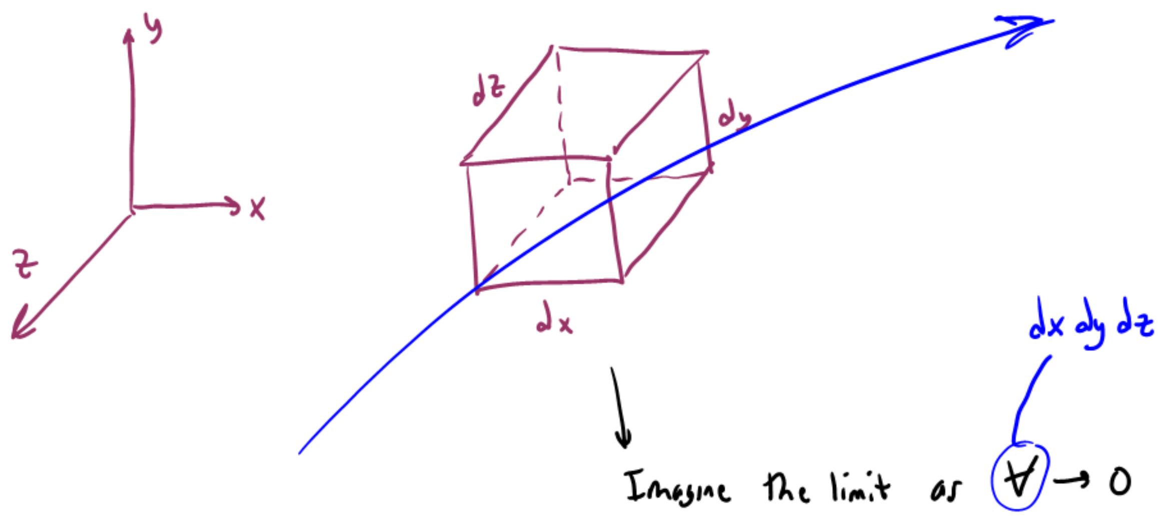
THE CONTINUITY EQUATION

In this lesson, we will:

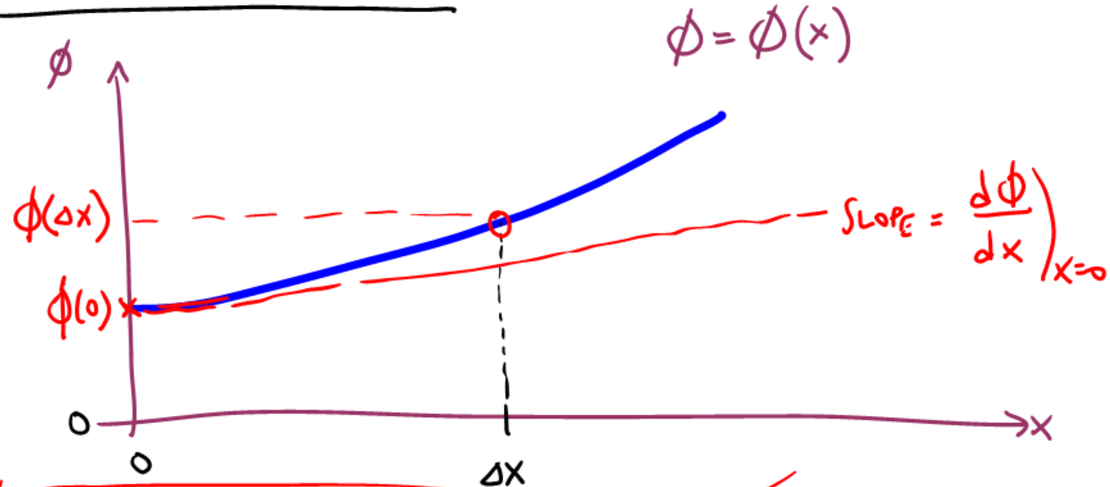
- Derive the **Continuity Equation** (the **Differential Equation for Conservation of Mass**)
- Discuss some **Simplifications** of this equation
- Do some example problems in both Cartesian and cylindrical coordinates

Derivation of the Continuity Equation

The derivation involves examination of the flow into and out of a tiny control volume that shrinks to zero volume in the limit. We utilize **Taylor Series Expansions**.



TAYLOR SERIES EXPANSIONS



$$\boxed{\phi(\Delta x) = \phi(0) + \frac{d\phi}{dx}\Delta x} + \cancel{\frac{1}{2!}\frac{d^2\phi}{dx^2}(\Delta x)^2} + \cdots$$

As $\Delta x \to 0$, the higher-order terms become negligible

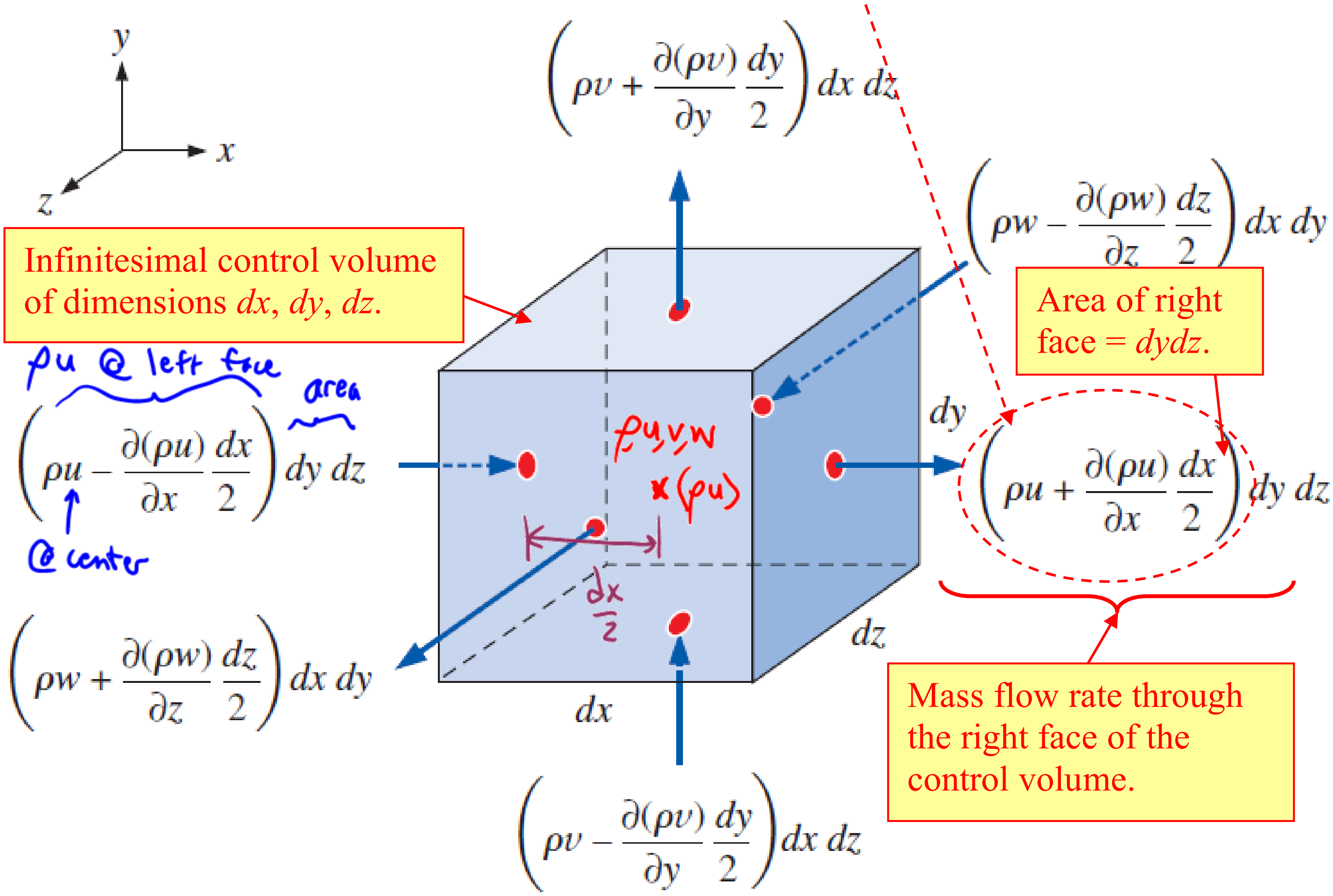
TRUNCATED TAYLOR SERIES (truncated to 1st-order)

Consider a tiny ***differential control volume***. First, we approximate the mass flow rate into or out of each of the six surfaces of the control volume, using ***Taylor series expansions*** around the center point, where the velocity components and density are u, v, w, and ρ. For example, at the right face,

Ignore terms higher than order dx.

$$(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}\left(\frac{dx}{2}\right)^2 + \cdots \qquad \textbf{(9–6)}$$

The mass flow rate through each face is equal to ρ times the normal component of velocity through the face times the area of the face. We show the mass flow rate through all six faces in the diagram below (Figure 9-5 in the text):



All copied figures and equations from Çengel and Cimbala, Ed. 4.

Next, we add up all the mass flow rates through all six faces of the control volume in order to generate the general (unsteady, incompressible) ***continuity equation***:

Net mass flow rate into CV: ← all the *positive* mass flow rates (into CV)

$$\sum_{\text{in}} \dot{m} \cong \underbrace{\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\text{left face}} + \underbrace{\left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz}_{\text{bottom face}} + \underbrace{\left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\, dy}_{\text{rear face}}$$

Net mass flow rate out of CV: ← all the *negative* mass flow rates (out of CV)

$$\sum_{\text{out}} \dot{m} \cong \underbrace{\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right) dy\, dz}_{\text{right face}} + \underbrace{\left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right) dx\, dz}_{\text{top face}} + \underbrace{\left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right) dx\, dy}_{\text{front face}}$$

We plug these into the integral conservation of mass equation for our control volume:

$$\boxed{\int_{\text{CV}} \frac{\partial \rho}{\partial t}\, dV} = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m} \qquad \textbf{(9–2)}$$

This term is approximated at the *center* of the tiny control volume, i.e.,

$$\int_{\text{CV}} \frac{\partial \rho}{\partial t}\, dV \cong \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

The conservation of mass equation (Eq. 9-2) thus becomes

$$\frac{\partial \rho}{\partial t} dx\, dy\, dz = -\frac{\partial(\rho u)}{\partial x} dx\, dy\, dz - \frac{\partial(\rho v)}{\partial y} dx\, dy\, dz - \frac{\partial(\rho w)}{\partial z} dx\, dy\, dz$$

Dividing through by the volume of the control volume, *dxdydz*, yields

Continuity equation in Cartesian coordinates:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0} \qquad \textbf{(9–8)}$$

Finally, we apply the definition of the ***divergence*** of a vector, i.e.,

$$\vec{\nabla} \cdot \vec{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} \quad \text{where} \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \quad \text{and} \quad \vec{G} = \left(G_x, G_y, G_z\right)$$

Letting $\vec{G} = \rho \vec{V}$ in the above equation, where $\vec{V} = (u, v, w)$, Eq. 9-8 is re-written as

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0 \qquad \textbf{(9–5)}$$

GENERAL VECTOR FORM OF THE CONTINUITY EQUATION

Simplifications

The above continuity equation is general – steady or unsteady, compressible or incompressible, valid for any coordinate system. Now let's consider some simplifications.

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0}$$

a. Steady but compressible

$\frac{\partial}{\partial t}(\ ) = 0$ Here $\frac{\partial \rho}{\partial t} = 0 \rightarrow \boxed{\vec{\nabla}\cdot(\rho\vec{V}) = 0}$ *

b. Incompressible but unsteady

$\rho = \text{constant}$ $\therefore \frac{\partial \rho}{\partial t} = 0 \rightarrow \vec{\nabla}\cdot(\rho\vec{V}) = 0$

$\cancel{\rho}(\vec{\nabla}\cdot\vec{V}) = 0$

$$\boxed{\vec{\nabla}\cdot\vec{V} = 0}$$ *

ρ has dropped out of the equation

- This eq. applies @ any instant in time
- The flow immediately (instantaneously) adjusts itself such that this eq. is satisfied.

SPEED OF SOUND

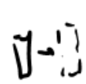
Compressible flow

Movement is sensed at some later time

c is finite

Incompressible flow

c is infinite



movement is felt instantaneously everywhere

c) Incompressible & steady flow

$$\boxed{\vec{\nabla} \cdot \vec{V} = 0}$$

Our "workhorse" eq.

INCOMPRESSIBLE CONTINUITY EQUATION

CARTESIAN COORDINATES (x, y, z) & (u, v, w)

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

CYLINDRICAL COORDINATES (r, θ, z) & (u_r, u_θ, u_z)

$$\boxed{\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0}$$

valid for either steady or unsteady incompressible flow

Example Problems

Example: Continuity equation

Given: A velocity field is given by

$$u = 3x + 4y \qquad v = by + 2x^2 \qquad w = 0$$

To do: Calculate *b* such that this a valid steady, incompressible velocity field.

Solution:

To be a valid steady, incompressible velocity field, it must satisfy continuity!

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$3 + b + 0 = 0$$

$$\boxed{b = -3}$$

Example: Continuity equation

Given: A velocity field is given by

$$u = ax + b \qquad v = \text{ unknown} \qquad w = 0$$

To do: Derive an expression for *v* so that this a valid steady, incompressible velocity field.

Solution:

To be a valid steady, incompressible velocity field, it must satisfy continuity!

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$a + \frac{\partial v}{\partial y} + 0 = 0$$

NOT A CONSTANT

$$\frac{\partial v}{\partial y} = -a \quad \longrightarrow \quad \text{INTEGRATE} \quad V = -ay + \cancel{C} + f(x,z)$$

$$\boxed{V = -ay + f(x,z)}$$

Verify:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \stackrel{?}{=} 0$$

$$a - a + 0 = 0 \checkmark \quad \text{YES}$$

Example: Continuity equation

Given: A flow field is 2-D in the r-θ plane, and its velocity field is given by

$$u_r = \text{ unknown} \qquad u_\theta = c\theta \qquad u_z = 0$$

To do: Derive an expression for u_r so that this a valid steady, incompressible velocity field.

Solution:

To be a valid steady, incompressible velocity field, it must satisfy continuity!

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) \quad + \quad \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \quad + \quad \frac{\partial u_z}{\partial z} \quad = 0$$

$$\cancel{\frac{1}{r}}\frac{\partial}{\partial r}(ru_r) + \cancel{\frac{1}{r}}c + 0 = 0$$

$$\frac{\partial}{\partial r}(ru_r) = -c \rightarrow \text{integrate:} \quad ru_r = -cr + f(\theta)$$

divide by r: $\boxed{u_r = -c + \frac{f(\theta)}{r}}$

Example: Continuity equation

Given: A flow field is 2-D in the r-θ plane, and its velocity field is given by

$$u_r = -\frac{3}{r} + 2 \qquad u_\theta = 2r + a\theta \qquad u_z = 0$$

To do: Calculate a such that this a valid steady, incompressible velocity field.

Solution:

To be a valid steady, incompressible velocity field, it must satisfy continuity!

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) \quad + \quad \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \quad + \quad \underbrace{\cancel{\frac{\partial u_z}{\partial z}}}_{0} \quad = 0$$

Mult by r: $\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_\theta}{\partial \theta} = 0$

$$\frac{\partial}{\partial r}(-3 + 2r) + a = 0$$

$$2 + a = 0 \rightarrow \boxed{a = -2}$$

Verify: $\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \frac{1}{r}(2) + \frac{1}{r}(-2) = 0$ ✓