STREAM FUNCTION, CARTESIAN COORDINATES

In this lesson, we will:

- Define the **Stream Function** and discuss its **Physical Significance**
- Discuss how to calculate the stream function and plot **Streamlines**
- Do some example problems

Definition of the Stream Function

ψ = stream function

Consider 2-D incompressible flow in the x-y plane

Continuity $\rightarrow$ $$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \quad (1)$$

DEFINE $$\boxed{u = \frac{\partial \psi}{\partial y}, \quad v = \frac{-\partial \psi}{\partial x}}$$

Plug into Eq. (1): $$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

$$\boxed{\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0}$$

$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$ for any smooth continuous function $\psi(x,y)$

Bottom Line:
- For a given 2-D incomp. flow field, if we can define $\psi(x,y)$, then continuity is exactly satisfied
- If we know $\psi(x,y)$, we can easily calculate u & v by the definition of ψ

Example: Stream Function in Cartesian Coordinates

Given: A flow field is 2-D in the *x*-*y* plane, and its stream function is given by

$$\psi(x,y) = ax^3 + byx$$

To do: Calculate the velocity components and verify that this stream function represents a valid steady, incompressible velocity field.

Solution: By definition of the stream function,

$$u = \frac{\partial \psi}{\partial y} = bx \qquad v = -\frac{\partial \psi}{\partial x} = -\left[3ax^2 + by\right]$$

$$\boxed{u = bx} \quad \boxed{v = -3ax^2 - by} \; *$$

To be a valid steady, incompressible velocity field, it must satisfy continuity!

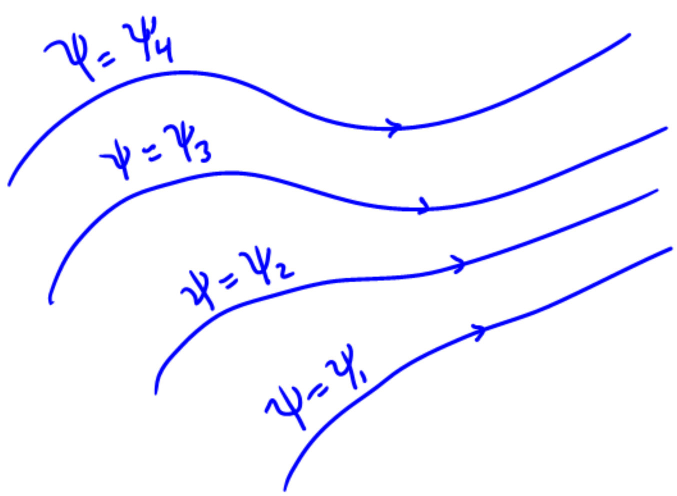
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$b \quad -b \quad + 0 = 0 \;\checkmark$$

Physical Significance of the Stream Function

a. Streamlines

- Curves of constant ψ are streamlines of the flow
- ψ = constant along a streamline

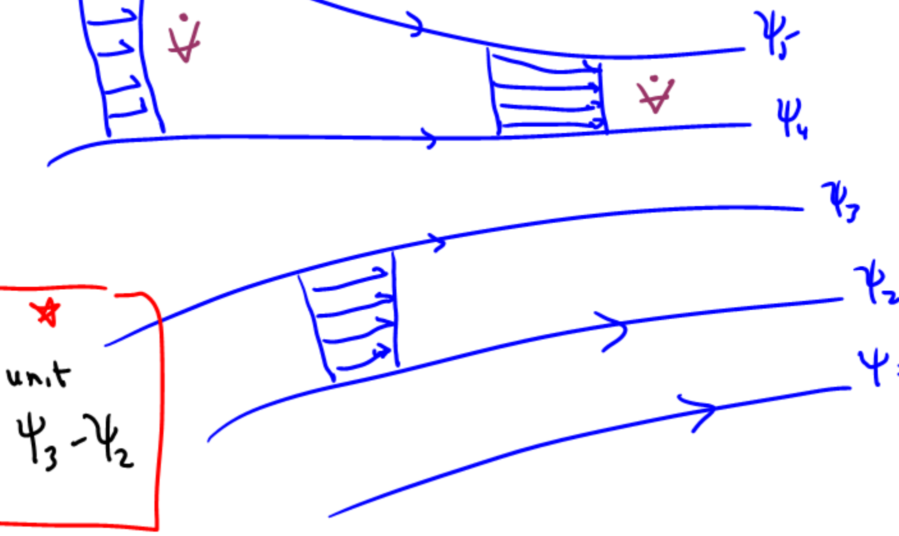


b. Volume Flow Rate

The difference in ψ from one streamline to another is equal to the volume flow rate per unit width between the two streamlines

(into the page)

$\dot{V}$ is constant between any two streamlines

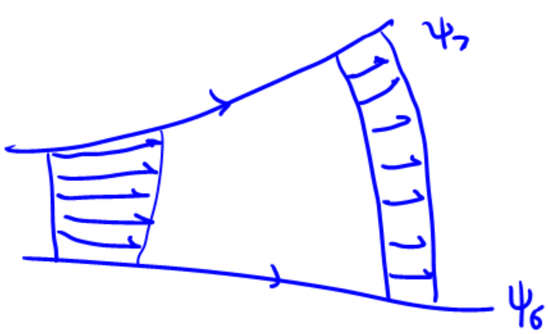


$\dot{V} = VA$

$\boxed{\dot{V}_{2\text{-}3} \text{ per unit depth} = \psi_3 - \psi_2}$

$\therefore$ As streamlines converge, the flow is accelerating

As streamlines diverge, the flow is decelerating



$\dot{V}$ per unit width $= \psi_7 - \psi_6$

Example: Plotting Streamlines Using the Stream Function

Given: A steady, 2-D, incompressible flow is given by this velocity field:

$$u = x^2 \qquad v = -2xy - 1 \qquad w = 0$$

To do: Generate an expression for stream function $\psi(x,y)$ and plot some streamlines.

Solution:

First, it is wise to verify that this is a valid steady, incompressible velocity field. If not, the problem is ill-posed and we could not continue. To verify, it must satisfy continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \checkmark$$

$$2x - 2x + 0 = 0 \quad \checkmark$$

- Pick one of the ψ definitions: $\frac{\partial \psi}{\partial y} = u$ or $\frac{\partial \psi}{\partial x} = -v$

I'll pick $\frac{\partial \psi}{\partial y} = u = x^2$

PARTIAL INTEGRATION

- Integrate w.r.t. y $\quad \boxed{\psi = x^2 y + f(x)}$ (2)
- Use the other ψ definition $\rightarrow \boxed{\frac{\partial \psi}{\partial x} = -v = 2xy + 1}$
- Equate & solve for ψ:

$$\frac{\partial \psi}{\partial x} = \underbrace{\boxed{2xy + f'(x)} = \boxed{2xy + 1}}_{f'(x) = 1}$$

total integral

integrate: $\boxed{f(x) = x + C}$

- Plug $f(x)$ into (2) $\rightarrow \boxed{\psi = x^2 y + x + C}$ ☆

ANSWER

Verify: ☺ ✓

$u = \frac{\partial \psi}{\partial y} = \underline{\underline{x^2}}$ $\qquad$ $v = -\frac{\partial \psi}{\partial x} = \underline{\underline{-2xy - 1}}$

Plot Streamlines

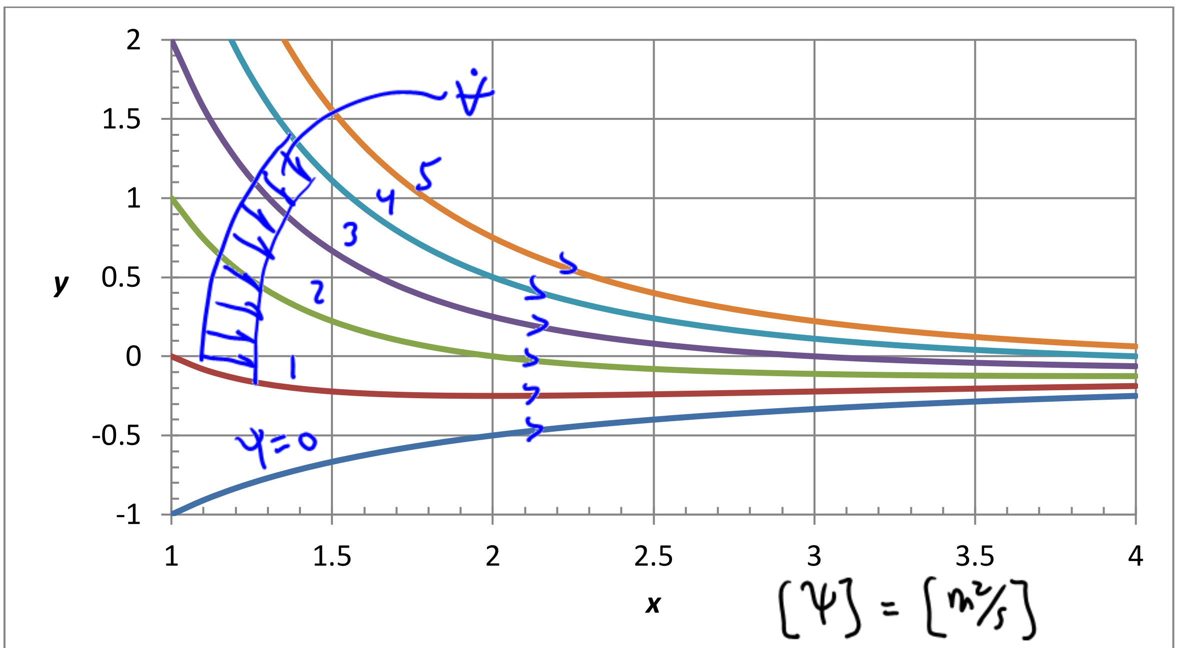
Solve for $y = \text{fnc}(x)$ @ various ψ values

• At $\psi = \psi_1 \rightarrow \psi_1 = x^2 y + x + C$

$$\boxed{y = \frac{\psi_1 - x - C}{x^2}}$$

☆ EQ. FOR STREAMLINE ψ_1

I chose $\underline{\underline{C=0}}$ in this example



$\{\psi\} = \{L^2/t\}$

Example: Streamlines and Volume Flow Rate

Given: The 2-D flow field shown above. The width (into the page) is $\underline{0.50 \text{ m.}} = b$

To do: Calculate the volume flow rate (in units of m^3/s) between streamlines $\psi = \underline{\underline{1}}\ m^2/s$ and $\psi = 4\ m^2/s$.

b = width into the page

Solution:

$\dot{\forall}$ per unit width $= \psi_4 - \psi_1$

$\frac{\dot{\forall}}{b}$

$$\boxed{\dot{\forall} = (\psi_4 - \psi_1) b}$$

$$\dot{\forall} = (4 - 1)\frac{m^2}{s}(0.50 \text{ m}) = \boxed{1.5\ m^3/s = \dot{\forall}}$$ ☆