THE IRROTATIONAL FLOW APPROXIMATION

In this lesson, we will:

- Define **Irrotational Regions of Flow** (also called **Potential Flow Regions**) and the equations of fluid motion that apply to such regions
- Introduce and define the **Velocity Potential Function** and how to apply it
- Discuss **Two-Dimensional Irrotational Flow** and its simplified equations
- Revisit the **Stream Function** and its application in 2-D irrotational regions of flow
- Show a simplified **Bernoulli Equation** for 2-D irrotational regions of flow
- Do an example problem

The Irrotational Flow Approximation and the Velocity Potential Function

Recall, Vorticity $= \vec{\zeta} = \vec{\nabla} \times \vec{V}$

- If $\vec{\zeta} = 0$, the flow is irrotational
- If $\vec{\zeta} \neq 0$, the flow is rotational

Vector Identity: $\boxed{\text{If } \vec{\nabla} \times \vec{B} = 0, \text{ then } \vec{B} = \vec{\nabla}\phi}$

ϕ is a scalar called the potential function

In fluid flow, let $\vec{B} = \vec{V}$

$\boxed{\text{If } \underbrace{\vec{\nabla} \times \vec{V}}_{\vec{\zeta}} = 0, \text{ then } \vec{V} = \vec{\nabla}\phi}$

ϕ = the velocity potential function

$\boxed{\text{If the flow is irrotational } (\vec{\zeta} = 0), \text{ then } \vec{V} = \vec{\nabla}\phi}$ ✱

IRROTATIONAL FLOW IS ALSO CALLED POTENTIAL FLOW

Components: $\boxed{\vec{V} = \vec{\nabla}\phi}$

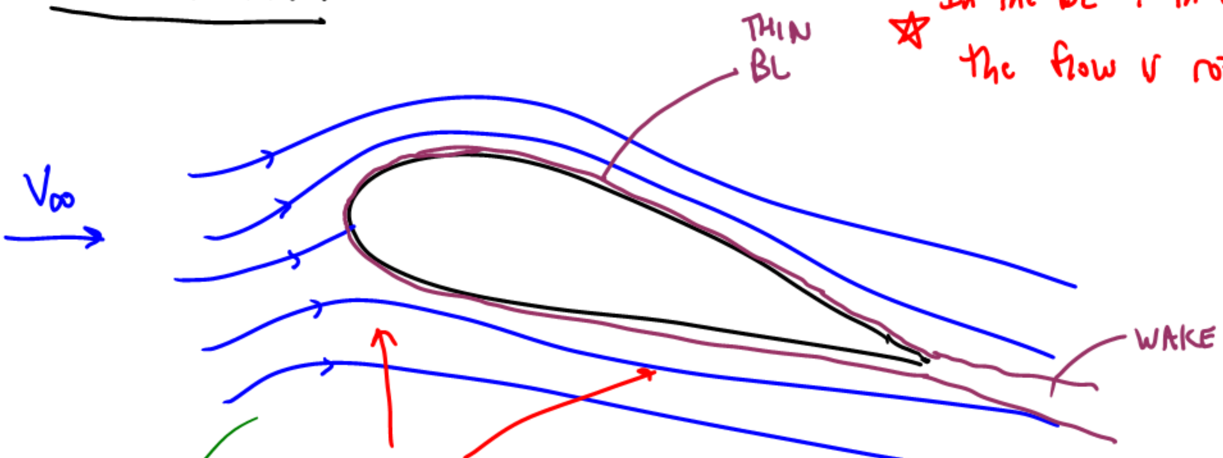
Cartesian: $\boxed{u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad w = \frac{\partial \phi}{\partial z}}$

ϕ is useful for 2-D or 3-D irrotational flow

Planar Cylindrical: $\boxed{u_r = \frac{\partial \phi}{\partial r} \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad u_z = \frac{\partial \phi}{\partial z}}$

APPLICABILITY:

☆ In the BL & in the wake, the flow is rotational



Outside of the BL & wake, the flow is ≈ irrotational

THESE ARE REGIONS OF IRROTATIONAL FLOW

EQUATIONS:

- continuity $\boxed{\vec{\nabla}\cdot\vec{V}=0}$ But $\vec{V}=\vec{\nabla}\phi$

$$\vec{\nabla}\cdot\vec{\nabla}\phi=0 \Rightarrow \boxed{\nabla^2\phi=0}$$

∇^2

Cartesian: (x,y,z)

$$\boxed{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0}$$

Planar Cylindrical: (r,θ,z)

$$\boxed{\nabla^2\phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0}$$

NAVIER-STOKES

$$\rho\frac{D\vec{V}}{Dt}=-\vec{\nabla}P+\rho\vec{g}+\mu\nabla^2\vec{V} \qquad \boxed{\vec{V}=\vec{\nabla}\phi}$$

$$\mu\nabla^2\vec{V} = \mu\nabla^2(\vec{\nabla}\phi) = \mu\vec{\nabla}(\nabla^2\phi)$$

order of differentiation does not matter if ϕ is smooth & continuous

But $\nabla^2 \phi = 0$ by continuity !

N-S equation reduces to $\boxed{\rho \dfrac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g}}$

EULER EQUATION

SAME EULER EQ. WE HAD FOR INVISCID FLOW

HERE, FOR IRROTATIONAL REGIONS OF FLOW, THE VISCOUS TERMS DROP OUT, BUT FOR A DIFFERENT REASON THAN FOR INVISCID REGIONS OF FLOW

Simplification of the Beloved Bernoulli Equation for Irrotational Flow Regions

From a previous lesson (inviscid flow approximation)

We started with the Euler Eq.,

used a vector identity & some algebra to get

$$\vec{\nabla}\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right) = \vec{V} \times \cancel{\vec{\zeta}}$$

$\vec{\zeta} = 0$ in an irrotational flow region

$$\vec{\nabla}\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right) = 0$$

We know that if $\vec{\nabla}(B) = 0$, then B must be a constant

Here,

$$\boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant everywhere}}$$

Valid for irrotational regions of flow

MOST BELOVED BERNOULLI EQUATION

Recall the "Beloved Bernoulli Equation"

$$\boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline}}$$

Valid for rotational flow (but for inviscid regions of flow)

The irrotational flow approximation is more restrictive than the inviscid flow approximation

- Inviscid regions of flow are not necessarily irrotational (e.g., solid body rotation)
- Irrotational regions of flow are not necessarily inviscid

Two-Dimensional Irrotational (Potential) Flow

We can model many interesting & practical flows if we restrict our analysis to flows that are

- 2-D
- steady
- incompressible
- irrotational (regions)

- Velocity potential $\vec{V} = \vec{\nabla}\phi \rightarrow \boxed{u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y}}$
- continuity $\boxed{\nabla^2 \phi = 0} \rightarrow \boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$
- Irrotationality $\vec{\zeta} = 0 \rightarrow \vec{\nabla} \times \vec{V} = 0 \rightarrow \boxed{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0}$
- stream function $\boxed{u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}}$

$$-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0$$
$$\parallel$$
$$\boxed{\nabla^2 \psi = 0} \; *$$

Summary of equations for 2-D, steady, incompressible, irrotational flow in the *x-y* plane, either Cartesian coordinates (x-y plane) or Cylindrical planar coordinates (*r-θ* plane):

$$\boxed{\vec{\zeta} = \vec{\nabla} \times \vec{V} = 0} \rightarrow \boxed{\vec{V} = \vec{\nabla}\phi} \rightarrow \boxed{\nabla^2 \phi = 0}; \quad \boxed{\nabla^2 \psi = 0} \;\&\; \boxed{\frac{P}{\rho} + \frac{V^2}{2} + gz = \text{constant everywhere}}$$

$V^2 = u^2 + v^2$

Cartesian: $\boxed{\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$, $\boxed{\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0}$,

$$\boxed{u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}}, \quad \boxed{v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}}, \quad \boxed{\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0}$$

Cylindrical: $\boxed{\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0}$, $\boxed{\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0}$,

$$\boxed{u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}}, \quad \boxed{u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}}, \quad \boxed{\zeta_z = \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} = 0}$$

Example: Irrotational Velocity and Pressure Fields

Given: A steady, 2-D, incompressible velocity field is given by velocity components,

$$u = 2x + y \qquad v = x - 2y$$

We are ignoring units in this problem (the units of the constants are adjusted such that the velocity components have appropriate units). We also ignore gravitational effects. At the origin $(x,y) = (0,0)$, $P = P_0$.

To do: Generate expressions for the velocity potential function, the stream function, and the pressure field.

Solution:

First verify our equations & approximations

Continuity $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 - 2 = 0$ ✓

Irrotationality: $\quad \zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 1 - 1 = 0$ ✓

THIS IS A VALID 2-D, STEADY, INCOMPRESSIBLE, IRROTATIONAL FLOW

Velocity potential: $\quad u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad [\vec{V} = \vec{\nabla}\phi]$

$$\frac{\partial \phi}{\partial x} = u = 2x + y$$

$$\phi = x^2 + xy + f(y)$$

$$\frac{\partial \phi}{\partial y} = x + f'(y) = v = x - 2y$$

$$f'(y) = -2y$$

$$f(y) = -y^2 + C_1$$

$$\boxed{\phi = x^2 + xy - y^2 + C_1}$$

- Verify Laplacian $\rightarrow \nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 - 2 = 0$ ✓

- Stream function $\rightarrow \frac{\partial \psi}{\partial y} = u = 2x + y \qquad \frac{\partial \psi}{\partial x} = -v = -x + 2y$

$$\psi = 2xy + \frac{y^2}{2} + f(x) \rightarrow \frac{\partial \psi}{\partial x} = 2y + f'(x)$$

$$f'(x) = -x \Rightarrow f(x) = -\frac{x^2}{2} + C_2$$

$$\boxed{\psi = -\frac{x^2}{2} + 2xy + \frac{y^2}{2} + C_2}$$

can verify that $\nabla^2\psi = 0$ ✓

Pressure: "Most Beloved Bernoulli Equation"

$$\frac{P}{\rho} + \frac{V^2}{2} + g\cancel{z} = \text{constant } \underline{\text{everywhere}} = \text{const}$$

no gravity here

mult by ρ & solve for $P \rightarrow P = \underbrace{\rho \cdot \text{const}}_{= C_3} - \rho\frac{V^2}{2}$

$$P = C_3 - \rho\frac{V^2}{2}$$

Apply BC: @ $(x,y) = (0,0)$, $P = P_0$

& $u = 2x + y = 0$
$v = x - 2y = 0$ $\Big\} V^2 = u^2 + v^2 = 0$

$P = P_0 = C_3 - 0$ @ the origin $\Rightarrow$ $\boxed{C_3 = P_0}$

$$\boxed{P = P_0 - \rho\frac{V^2}{2}}$$ ☆