RAYLEIGH FLOW – COMPRESSIBLE FLOW WITH HEAT TRANSFER

In this lesson, we will:

- Introduce **Rayleigh flow**: flow in a duct with ***heat addition*** (or removal) but no friction
- Discuss Rayleigh flow qualitatively and quantitatively
- Do an example problem

Disclaimer: This is an ***abbreviated*** summary of Rayleigh flow; a more rigorous analysis is presented in my compressible flow course (ME 420 at Penn State University)

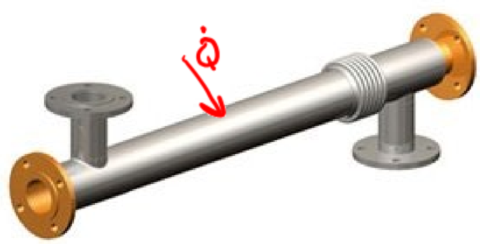
Rayleigh Flow Introduction, Approximations, and Assumptions

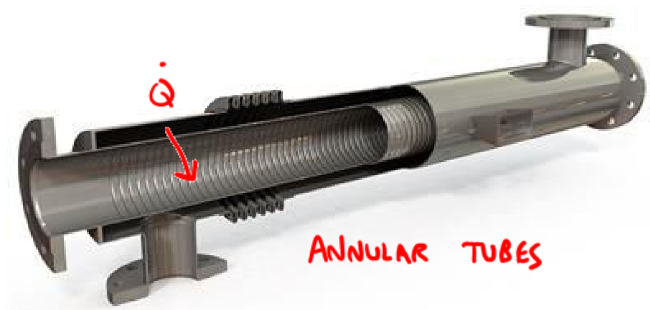
- Steady
- One-D flow (ignore boundary layers) such that V is approximately constant at any cross-section of the duct, i.e., $V = V(x)$ only

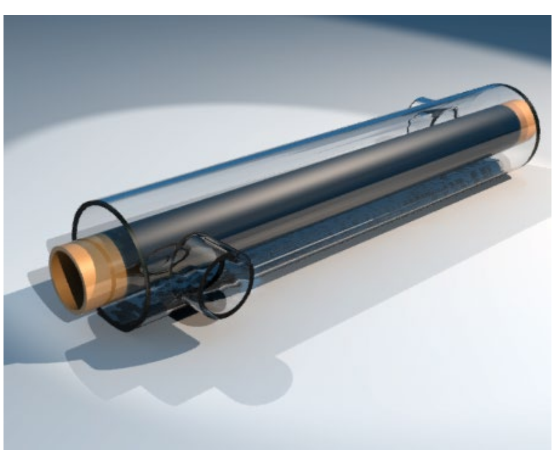
- Ideal gas
- Constant area duct (straight section of pipe)
- Constant gas properties (k, c_P, R, etc.) *even if chemical reactions or combustion provides the heat input* (different gas properties of the combustion products and/or different mixture of gases after a reaction) enables analytic solution
- Negligible friction along duct walls

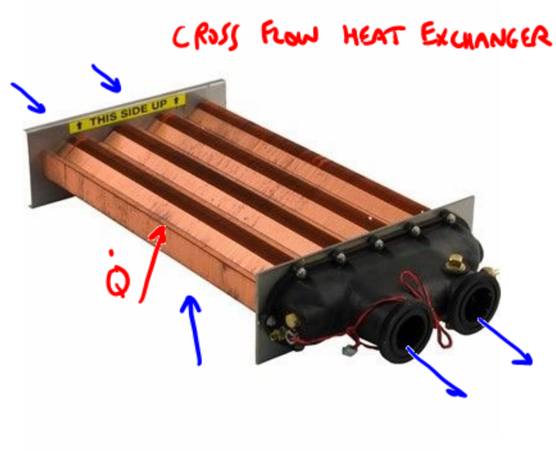
Applications of Rayleigh Flow

Gas heat exchangers:



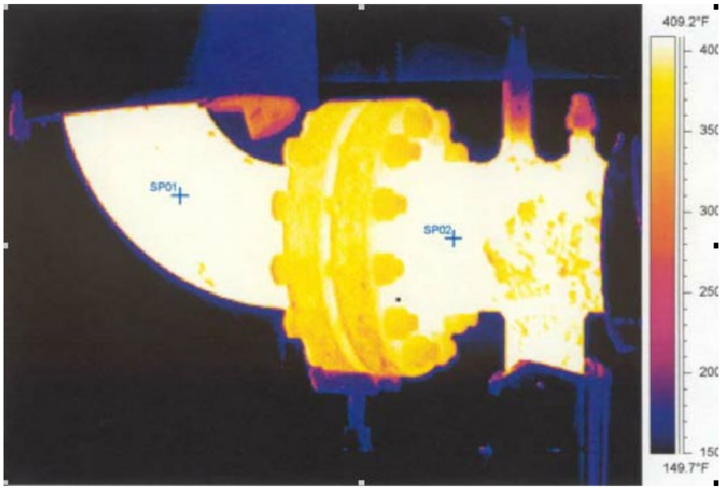


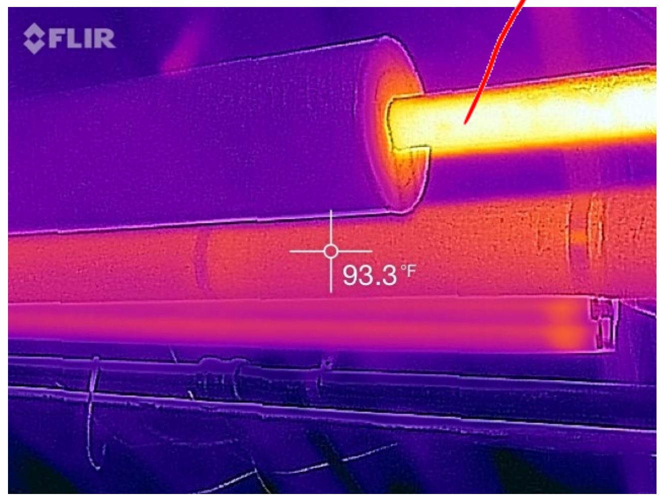




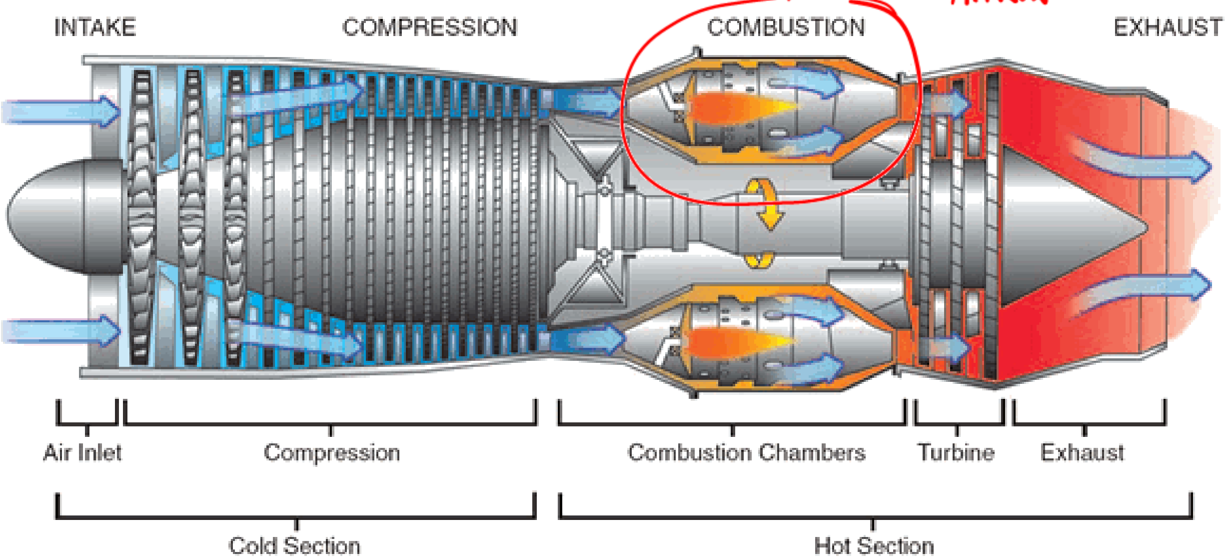
Gas flowing in uninsulated pipes (e.g., steam pipes): THERMAL IMAGES

$\dot{Q}$

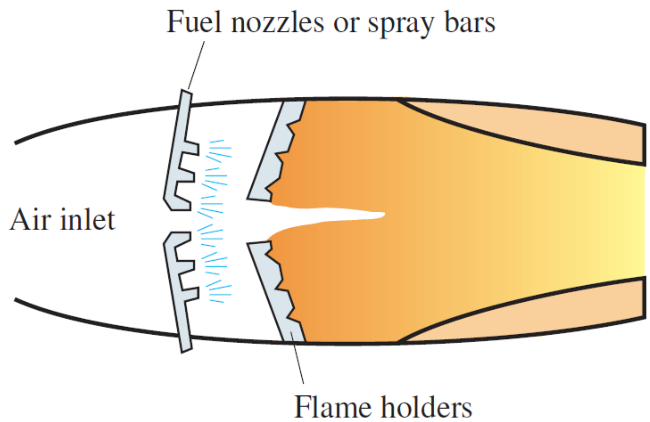




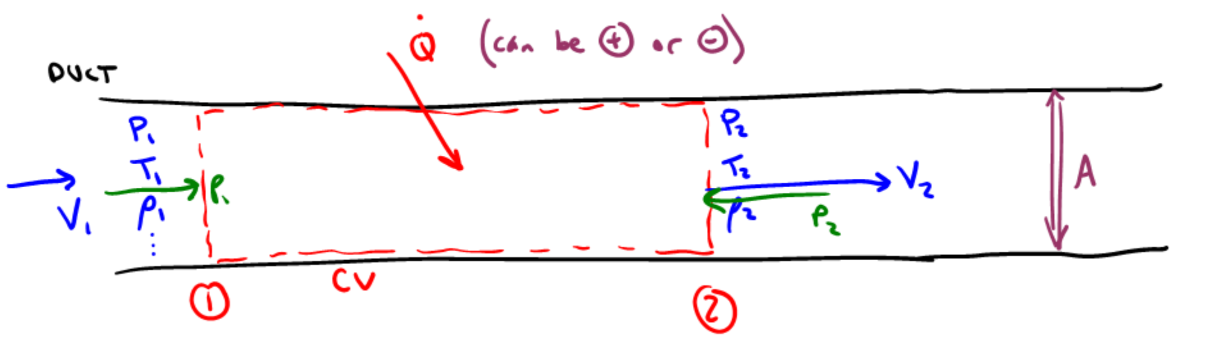
Combustors in a jet engine or gas turbine engine: RAYLEIGH FLOW APPROX



Afterburners in a jet engine:



Control Volume Analysis and the Rayleigh Curve



• Cons. of mass $\qquad \dot{m}_1 = \dot{m}_2 \rightarrow \rho_1 V_1 \cancel{A} = \rho_2 V_2 \cancel{A}$

$$\boxed{\rho_1 V_1 = \rho_2 V_2} \quad (1)$$

Linear momentum equation in x-direction:

$$\sum F_x = \cancel{\sum F_{x,\text{gravity}}} + \sum F_{x,\text{pressure}} + \cancel{\sum F_{x,\text{viscous}}} + \cancel{\sum F_{x,\text{other}}} = \sum_{\text{out}} \beta \dot{m} V - \sum_{\text{in}} \beta \dot{m} V$$

none in x-dir   ignore   none $\beta = 1$

$$\rho_2 V_2 A V_2 - \rho_1 V_1 A V_1$$

$$P_1 \cancel{A} - P_2 \cancel{A} = \rho_2 V_2^2 \cancel{A} - \rho_1 V_1^2 \cancel{A}$$

$$\boxed{P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2} \quad (2)$$

- EQs (1) & (2) must be satisfied for Rayleigh flow regardless of $\dot{Q}$

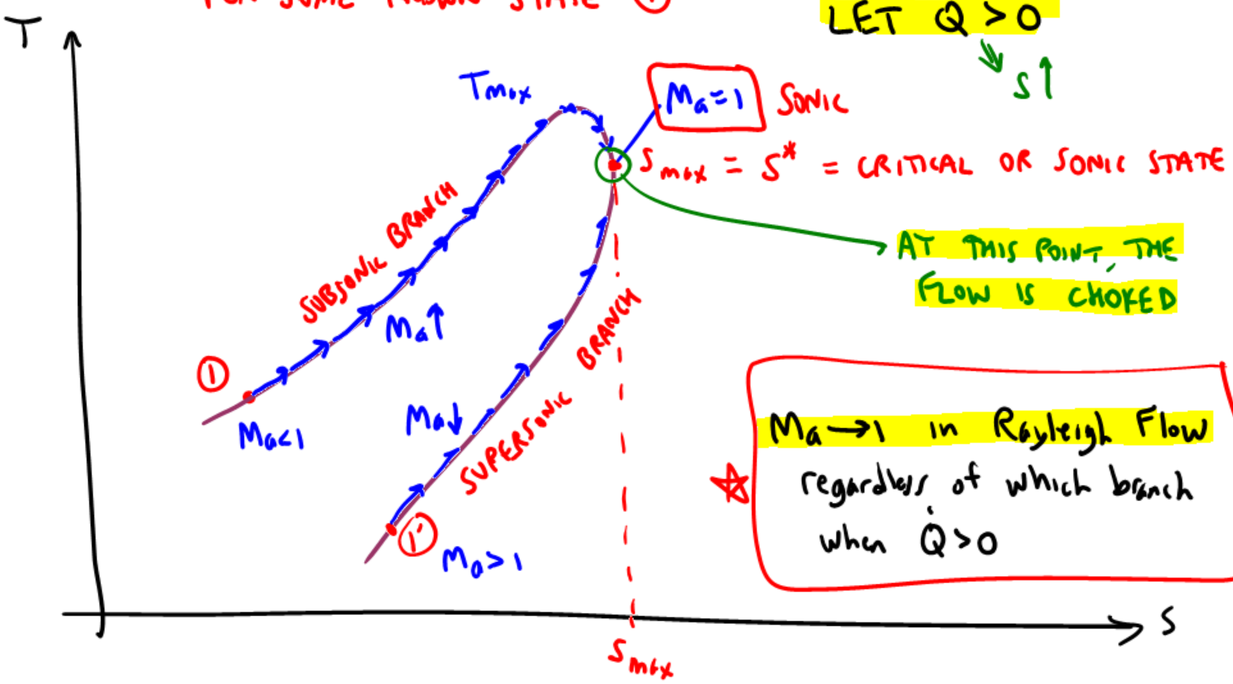
Algebra → · Eq. (1), (2)
· Ideal gas law
· T-ds eqs
· Compressible flow eqs

e.g., $\frac{T_0}{T} = 1 + \frac{k-1}{2} Ma^2$

$\frac{P_0}{P} =$ } etc.

WE GENERATE THE RAYLEIGH CURVE (or RAYLEIGH LINE) (ON A T-S diagram)
FOR SOME KNOWN STATE 1

LET $\dot{Q} > 0$ → $s\uparrow$



★ $Ma \rightarrow 1$ in Rayleigh Flow regardless of which branch when $\dot{Q} > 0$

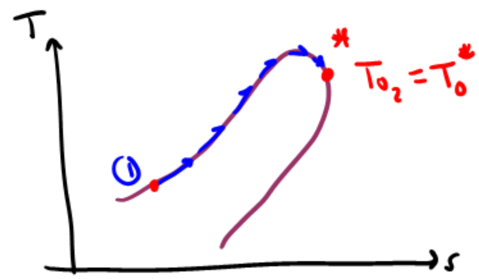
• **CONS. OF ENERGY** → Reduces to $\dot{Q} = \dot{m}\underbrace{\left(h_2 + \frac{V_2^2}{2}\right)}_{h_{0_2}} - \dot{m}\underbrace{\left(h_1 + \frac{V_1^2}{2}\right)}_{h_{0_1}}$

$\div \dot{m} \Rightarrow \quad \dfrac{\dot{Q}}{\dot{m}} = q = h_{0_2} - h_{0_1}$

IDEAL GAS $\quad h = C_p T$

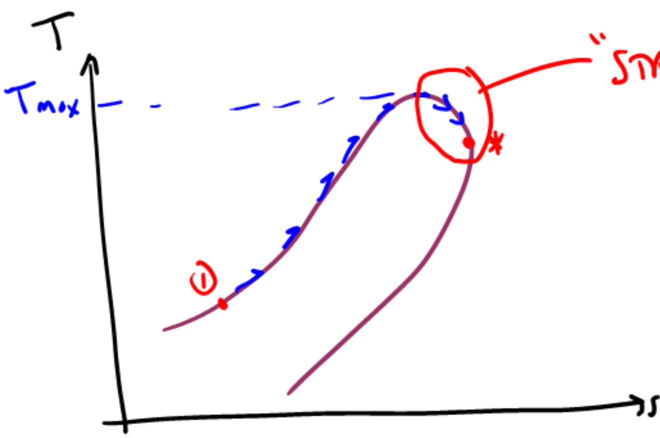
$h_0 = C_p T_0$

$$☆ \quad \boxed{q = C_p\left(T_{0_2} - T_{0_1}\right)} \quad (3)$$



$$@\ * \quad \boxed{q = q_{max} = C_p\left(T_0^* - T_{0_1}\right)}$$

MAX HEAT TRANSFER W/O CHANGING ①



"STRANGE ZONE"

T↓ as we add more heat!

Summary of Equations for Rayleigh Flow for an Ideal Gas

Conservation laws of mass, momentum, and energy (from above notes):

$$\rho_1 V_1 = \rho_2 V_2 \qquad P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2 \qquad q = \frac{\dot{Q}}{\dot{m}} = c_P\left(T_{02} - T_{01}\right)$$

Maximum possible heat transfer: $q_{\max} = \dfrac{\dot{Q}_{\max}}{\dot{m}} = c_P\left(T_0^* - T_{01}\right)$

Static property ratios (from more algebra, not shown):

$$\frac{P_2}{P_1} = \frac{1 + k\mathrm{Ma}_1^2}{1 + k\mathrm{Ma}_2^2} \rightarrow \frac{P}{P^*} = \frac{1 + k}{1 + k\mathrm{Ma}^2} \qquad \frac{T_2}{T_1} = \left[\frac{\mathrm{Ma}_2}{\mathrm{Ma}_1}\frac{1 + k\mathrm{Ma}_1^2}{1 + k\mathrm{Ma}_2^2}\right]^2 \rightarrow \frac{T}{T^*} = \left[\frac{(1+k)\mathrm{Ma}}{1 + k\mathrm{Ma}^2}\right]^2$$

$$\frac{\rho^*}{\rho} = \frac{V}{V^*} = \frac{(1+k)\mathrm{Ma}^2}{1 + k\mathrm{Ma}^2}$$

Stagnation property ratios (from more algebra, not shown):

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}\mathrm{Ma}^2 \qquad \frac{T_0}{T_0^*} = \frac{\left[2 + (k-1)\mathrm{Ma}^2\right](1+k)\mathrm{Ma}^2}{\left[1 + k\mathrm{Ma}^2\right]^2} \qquad \frac{P_0}{P_0^*} = \left[\frac{2 + (k-1)\mathrm{Ma}^2}{1+k}\right]^{\frac{k}{k-1}} \frac{(1+k)}{1 + k\mathrm{Ma}^2}$$

Step-by-Step Procedure to Solve Rayleigh Flow Problems

1. For known conditions at 1 and known rate of heat transfer, use $q = \dfrac{\dot{Q}}{\dot{m}} = c_P\left(T_{02} - T_{01}\right)$ to calculate T_{02}.
   $\dot{Q}$
2. Calculate T_{01}/T_0^* from the ratio equation: $\dfrac{T_{01}}{T_0^*} = \dfrac{\left[2 + (k-1)\mathrm{Ma}_1^2\right](1+k)\mathrm{Ma}_1^2}{\left[1 + k\mathrm{Ma}_1^2\right]^2}$.
3. Calculate T_{02}/T_0^* from clever use of ratios: $\dfrac{T_{02}}{T_0^*} = \dfrac{T_{02}}{T_{01}}\dfrac{T_{01}}{T_0^*}$.
4. Use the ratio equation for stagnation temperature (inversely) to calculate Ma_2:

$$\frac{T_{02}}{T_0^*} = \frac{\left[2 + (k-1)\mathrm{Ma}_2^2\right](1+k)\mathrm{Ma}_2^2}{\left[1 + k\mathrm{Ma}_2^2\right]^2}$$

→ IMPLICIT EQ → Solve for Ma_2 iteratively

[I typically use the False Position Method]

5. Use the remaining Rayleigh flow equations, ideal gas law, speed of sound equation, etc. to calculate other desired properties at State 2, such as T_2, P_2, V_2, c_2, h_2, ρ_2, etc.

Example: Rayleigh flow

Given:

- Air and fuel enter a 15-cm diameter tube at 550 K, 480 kPa, and 80.0 m/s. $= V$ (1)
- The fuel is burned between locations 1 and 2 in the tube, as sketched, and in the process, 4514 kW of heat is added. $\dot{Q}$

To do: Estimate the temperature, pressure, velocity, and Mach number at location 2.

Solution:

Assumptions and Approximations (consistent with our simplified Rayleigh flow analysis):

1. The air/fuel mixture is an ideal gas with the same properties as air alone, and the properties do not change due to combustion products.
2. The flow is steady and one-D.
3. Friction along the tube walls is negligible.

Inlet conditions and rate of heat transfer

Given: $V_1 = 80.0$ m/s, $T_1 = 550$ K, $P_1 = 480$ kPa, $A = \pi D^2/4 = 0.017671\ \text{m}^2$, $\dot{Q} = 4514$ kW

$\rho_1 = \dfrac{P_1}{RT_1}$ $\quad c_1 = \sqrt{kRT_1}$ $\quad \mathrm{Ma}_1 = \dfrac{V_1}{c_1}$

$\rho_1 = 3.0409\ \text{kg/m}^3$, $\quad c_1 = 470.10$ m/s, $\quad \mathrm{Ma}_1 = 0.17018$

$$T_{01} = \left(\frac{T_{01}}{T_1}\right)T_1 = \left(1 + \frac{k-1}{2}\mathrm{Ma}_1^{\,2}\right)T_1 = \left(1 + \frac{1.4-1}{2}(0.17018)^2\right)(550\ \text{K}) = \boxed{553.19\ \text{K} = T_{01}}$$

Step 1: Heat transfer analysis to calculate T_{02}

$$\dot{m}_1 = \rho_1 V_1 A_1 = \dot{m}_2 = \dot{m} = \left(3.0409\ \frac{\text{kg}}{\text{m}^3}\right)\left(80.0\ \frac{\text{m}}{\text{s}}\right)\left(0.017671\ \text{m}^2\right) = \boxed{4.2989\ \frac{\text{kg}}{\text{s}} = \dot{m}}$$

$$q = \frac{\dot{Q}}{\dot{m}} = \frac{4514\ \cancel{\text{kW}}}{4.2989\ \text{kg}/\cancel{\text{s}}}\left(\frac{\text{kJ}}{\cancel{\text{s}}\cdot\cancel{\text{kW}}}\right) = \boxed{1050.0\ \frac{\text{kJ}}{\text{kg}} = q}$$

$$T_{02} = T_{01} + \frac{q}{c_P} = 553.19\ \text{K} + \frac{1050.0\ \text{kJ/kg}}{1.0045\ \text{kJ/kg K}} = \boxed{1598.5\ \text{K} = T_{02}}$$

Step 2: Sonic (critical or *) reference values analysis to calculate T_{01}/T_0^*

$$\frac{T_{01}}{T_0^*} = \frac{\left[2 + (k-1)\right)\mathrm{Ma}_1^{\,2}\left](1+k)\mathrm{Ma}_1^{\,2}}{\left[1 + k\mathrm{Ma}_1^{\,2}\right]^2} = 0.12914 = \frac{T_{01}}{T_0^*} \rightarrow \boxed{T_0^* = 4283.7\ \text{K}}$$

Step 3: Calculate T_{02}/T_0^* from clever use of ratios

$$\frac{T_{02}}{T_0^*}=\frac{T_{02}}{T_{01}}\frac{T_{01}}{T_0^*}=\left(\frac{1598.5\text{ K}}{553.19\text{ K}}\right)(0.12914)=\boxed{0.37315=\frac{T_{02}}{T_0^*}}$$

Step 4: Calculate Ma_2 (inversely); I used the **False Position Method**

$$\frac{T_{02}}{T_0^*}=\frac{\left[2+(k-1)\text{Ma}_2^{\ 2}\right](1+k)\text{Ma}_2^{\ 2}}{\left[1+k\text{Ma}_2^{\ 2}\right]^2}=0.37315\ \rightarrow\ \boxed{\text{Ma}_2=0.31429}$$

STILL SUBSONIC!

Step 5: Calculate other desired properties at State 2

- Verify that $\dot{Q}<\dot{Q}_{max}$ since not choked ✓

$$\dot{Q}_{max}=\dot{m}c_p\left(T_0^*-T_{01}\right)$$

$$=\left(4.2989\ \frac{\text{kg}}{\text{s}}\right)\left(1.0045\ \frac{\text{kJ}}{\text{kg}\cdot\text{K}}\right)(4283.7-553.19)\text{K}\left(\frac{\text{s}\cdot\text{kW}}{\text{kJ}}\right)$$

$$\dot{Q}_{max}=16109\text{ kW}$$

$\dot{Q}_{max}>$ our actual $\dot{Q}$ (4514 kW)

- $T_2\rightarrow\ \dfrac{T_{02}}{T_0^*}=0.37315$ & $\text{Ma}_2=0.31429$

$$T_{02}=(0.37315)(4283.7\text{ K})=1598.5\text{ K}=T_{02}$$

$$\frac{T_{02}}{T_2}=1+\frac{k-1}{2}\text{Ma}_2^{\ 2}\ \rightarrow\ \boxed{T_2=1567.6\text{ K}}$$

- Similarly $\rightarrow$ calc P_2, ρ_2, V_2, …

FINAL ANSWERS TO 3 DIGITS

$$\boxed{\begin{aligned}T_2&=1570\text{ K}\\P_2&=439\text{ kPa}\\V_2&=249\text{ m/s}\\\text{Ma}_2&=0.314\end{aligned}}$$