Angular Momentum Control Volume Analysis
(Section 6-6, Çengel and Cimbala)

1. Equations and definitions

See the derivation in the book, using the Reynolds transport theorem. The result is:

General:
$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho\, dV + \int_{CS} (\vec{r} \times \vec{V})\rho(\vec{V}_r \cdot \vec{n})\, dA \qquad (6\text{–}47)$$

(Relative velocity)

which can be stated as

$$\begin{pmatrix}\text{The sum of all}\\ \text{external moments}\\ \text{acting on a CV}\end{pmatrix} = \begin{pmatrix}\text{The time rate of change}\\ \text{of the angular momentum}\\ \text{of the contents of the CV}\end{pmatrix} + \begin{pmatrix}\text{The net flow rate of}\\ \text{angular momentum}\\ \text{out of the control}\\ \text{surface by mass flow}\end{pmatrix}$$

We simplify the control surface integral for cases in which there are well-defined inlets and outlets, just as we did previously for mass, energy, and momentum. The result is:

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho\, dV + \sum_{out} \vec{r} \times \dot{m}\vec{V} - \sum_{in} \vec{r} \times \dot{m}\vec{V} \qquad (6\text{–}50)$$

Note that we cannot define an "angular momentum flux correction factor" like we did previously for the kinetic energy and momentum flux terms. Furthermore, many problems we consider in this course are *steady*. For steady flow, Eq. 6-50 reduces to:

Steady flow:
$$\sum \vec{M} = \sum_{out} \vec{r} \times \dot{m}\vec{V} - \sum_{in} \vec{r} \times \dot{m}\vec{V} \qquad (6\text{–}51)$$

Net moment or torque acting on the control volume by external means = Rate of flow of angular momentum *out* of the control volume by mass flow − Rate of flow of angular momentum *into* the control volume by mass flow

Finally, in many cases, we are concerned about only *one* axis of rotation, and we simplify Eq. 6-51 to a scalar equation,

$$\sum M = \sum_{out} r\dot{m}V - \sum_{in} r\dot{m}V \qquad (6\text{–}52)$$

Equation 6-52 is the form of the angular momentum control volume equation that we will most often use, noting that r is the shortest distance (i.e. the *normal* distance) between the point about which moments are taken and the *line of action* of the force or velocity being considered. By convention, *counterclockwise moments are positive*.

2. Examples

See Examples 6-8 and 6-9 in the book. Example 6-9 is discussed in more detail here.

EXAMPLE 6–9 Power Generation from a Sprinkler System

A large lawn sprinkler with four identical arms is to be converted into a turbine to generate electric power by attaching a generator to its rotating head, as shown in Fig. 6–38. Water enters the sprinkler from the base along the axis of rotation at a rate of 20 L/s and leaves the nozzles in the tangential direction. The sprinkler rotates at a rate of 300 rpm in a horizontal plane. The diameter of each jet is 1 cm, and the normal distance between the axis of rotation and the center of each nozzle is 0.6 m. Estimate the electric power produced.

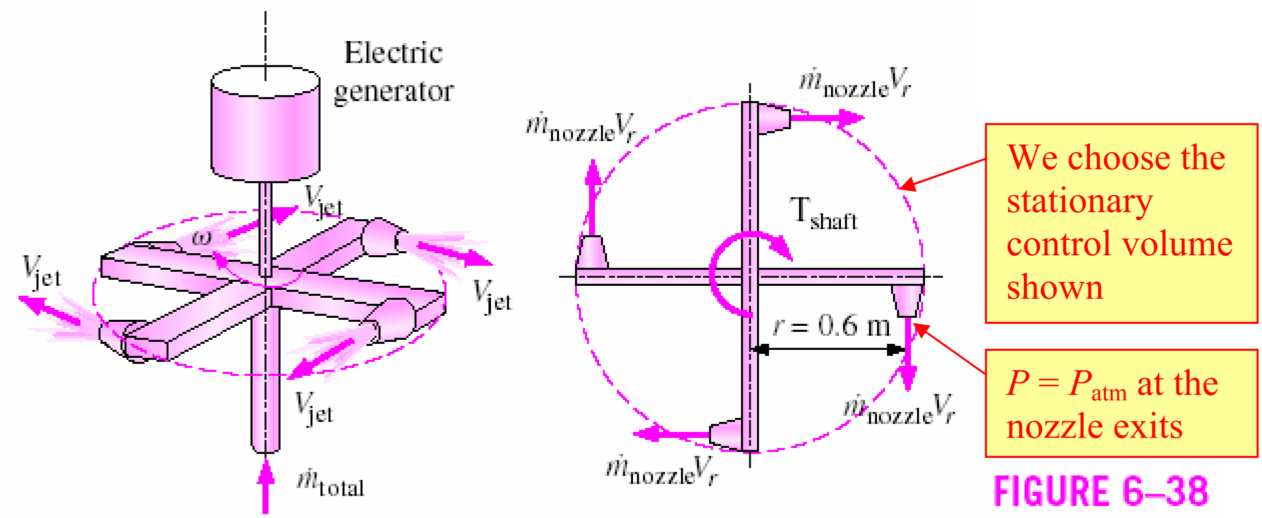

FIGURE 6–38

Assumptions **1** The flow is cyclically steady (i.e., steady from a frame of reference rotating with the sprinkler head). **2** The water is discharged to the atmosphere, and thus the gage pressure at the nozzle exit is zero. **3** Generator losses and air drag of rotating components are neglected. **4** The nozzle diameter is small compared to the moment arm, and thus we use average values of radius and velocity at the outlet.

Properties We take the density of water to be 1000 kg/m^3 = 1 kg/L.

Analysis We take the disk that encloses the sprinkler arms as the control volume, which is a stationary control volume.

Conservation of mass:

The conservation of mass equation for this steady-flow system is $\dot{m}_1 = \dot{m}_2 = \dot{m}_{total}$. Noting that the four nozzles are identical, we have $\dot{m}_{nozzle} = \dot{m}_{total}/4$ or $\dot{V}_{nozzle} = \dot{V}_{total}/4$ since the density of water is constant. The average jet exit velocity relative to the nozzle is

$$V_{jet} = \frac{\dot{V}_{nozzle}}{A_{jet}} = \frac{5 \text{ L/s}}{[\pi(0.01 \text{ m})^2/4]}\left(\frac{1 \text{ m}^3}{1000 \text{ L}}\right) = 63.66 \text{ m/s}$$

The angular and tangential velocities of the nozzles are

$$\omega = 2\pi\dot{n} = 2\pi(300 \text{ rev/min})\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 31.42 \text{ rad/s}$$

$$V_{\text{nozzle}} = r\omega = (0.6 \text{ m})(31.42 \text{ rad/s}) = 18.85 \text{ m/s}$$

That is, the water in the nozzle is also moving at a velocity of 18.85 m/s in the opposite direction when it is discharged. Then the average velocity of the water jet relative to the control volume (or relative to a fixed location on earth) becomes

$$V_r = V_{\text{jet}} - V_{\text{nozzle}} = 63.66 - 18.85 = 44.81 \text{ m/s}$$

Note: We must use the velocity relative to the control volume, which in this case is the *absolute* velocity, since our control volume is fixed (not moving).

Conservation of angular momentum:

Noting that this is a cyclically steady-flow problem, and all forces and momentum flows are in the same plane, the angular momentum equation can be approximated as $\sum M = \sum_{\text{out}} r\dot{m}V - \sum_{\text{in}} r\dot{m}V$, where r is the moment arm, all moments in the counterclockwise direction are positive, and all moments in the clockwise direction are negative.

The free-body diagram of the disk that contains the sprinkler arms is given in Fig. 6–38. Note that the moments of all forces and momentum flows passing through the axis of rotation are zero. The momentum flows via the water jets leaving the nozzles yield a moment in the clockwise direction and the effect of the generator on the control volume is a moment also in the clockwise direction (thus both are negative). Then the angular momentum equation about the axis of rotation becomes

Be careful with the *signs*.

$$-\text{T}_{\text{shaft}} = -4r\dot{m}_{\text{nozzle}}V_r \quad \text{or} \quad \text{T}_{\text{shaft}} = r\dot{m}_{\text{total}}V_r$$

Substituting, the torque transmitted through the shaft is determined to be

$$\text{T}_{\text{shaft}} = r\dot{m}_{\text{total}}V_r = (0.6 \text{ m})(20 \text{ kg/s})(44.81 \text{ m/s})\left(\frac{1 \text{ N}}{1 \text{ kg}\cdot\text{m/s}^2}\right) = 537.7 \text{ N}\cdot\text{m}$$

since $\dot{m}_{\text{total}} = \rho\dot{V}_{\text{total}} = (1 \text{ kg/L})(20 \text{ L/s}) = 20 \text{ kg/s}$.

Calculation of the shaft power:

Then the ~~power generated~~ Shaft power becomes

$$\dot{W} = 2\pi\dot{n}T_{\text{shaft}} = \omega T_{\text{shaft}} = (31.42 \text{ rad/s})(537.7 \text{ N}\cdot\text{m})\left(\frac{1 \text{ kW}}{1000 \text{ N}\cdot\text{m/s}}\right) = \mathbf{16.9 \text{ kW}}$$

Therefore, this sprinkler-type turbine has the potential to produce 16.9 kW of power.

The actual power generated will be less than this because of generator inefficiencies. We can calculate the generated power as

$$\dot{W}_{\text{electric}} = \eta_{\text{generator}}\dot{W}_{\text{shaft}}$$

Discussion To put the result obtained in perspective, we consider two limiting cases. In the first limiting case, the sprinkler is stuck and thus the angular velocity is zero. The torque developed will be maximum in this case since $V_{\text{nozzle}} = 0$ and thus $V_r = V_{\text{jet}} = 63.66$ m/s, giving $T_{\text{shaft, max}} = 764$ N · m. But the power generated will be zero since the shaft does not rotate.

In the second limiting case, the shaft is disconnected from the generator (and thus both the torque and power generation are zero) and the shaft accelerates until it reaches an equilibrium velocity. Setting $T_{\text{shaft}} = 0$ in the angular momentum equation gives $V_r = 0$ and thus $V_{\text{jet}} = V_{\text{nozzle}} = 63.66$ m/s. The corresponding angular speed of the sprinkler is

$$\dot{n} = \frac{\omega}{2\pi} = \frac{V_{\text{nozzle}}}{2\pi r} = \frac{63.66 \text{ m/s}}{2\pi(0.6 \text{ m})}\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 1013 \text{ rpm}$$

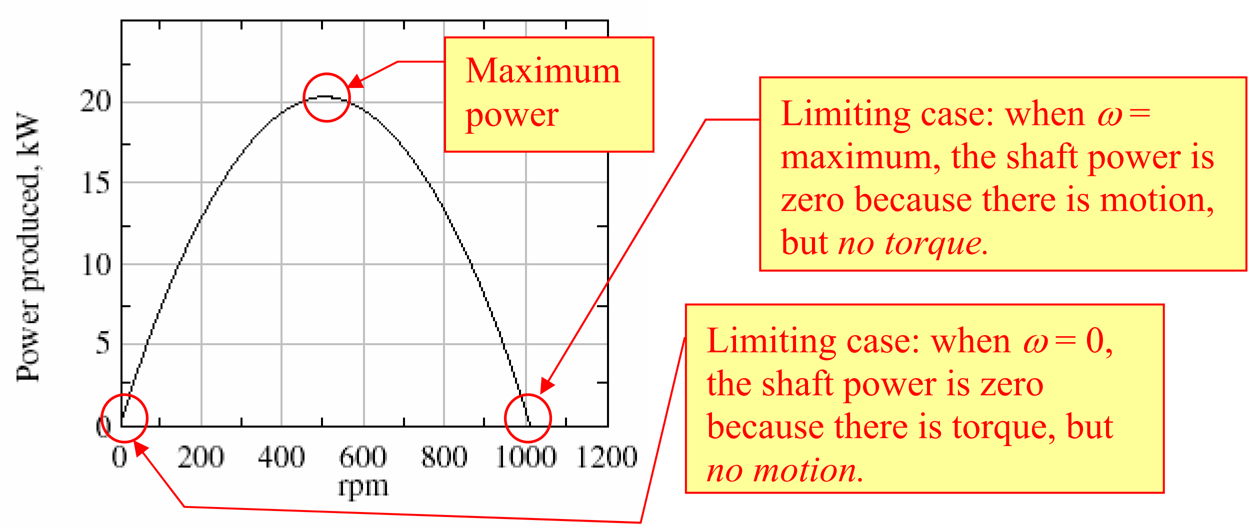

FIGURE 6–39

The variation of power produced with angular speed.