Derivation of the Navier-Stokes Equation
(Section 9-5, Çengel and Cimbala)

We begin with the general differential equation for conservation of linear momentum, i.e., ***Cauchy's equation***, which is valid for any kind of fluid,

Cauchy's equation: $$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} + \vec{\nabla}\cdot\sigma_{ij} \qquad \text{(9–50)}$$ Stress tensor

The problem is that the stress tensor σ_{ij} needs to be written in terms of the primary unknowns in the problem in order for Cauchy's equation to be useful to us. The equations that relate σ_{ij} to other variables in the problem – velocity, pressure, and fluid properties – are called ***constitutive equations***. There are different constitutive equations for different kinds of fluids.

Types of fluids:

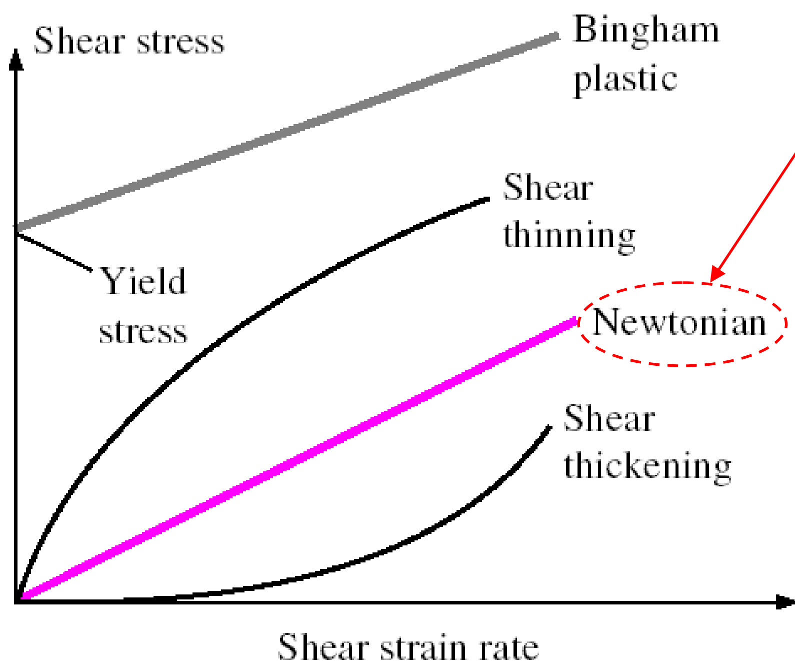

FIGURE 9–37

Rheological behavior of fluids—shear stress as a function of shear strain rate.

*For **Newtonian fluids**, the shear stress is linearly proportional to the shear strain rate.*

Examples of Newtonian fluids: water, air, oil, gasoline, most other common fluids.

Some examples of non-Newtonian fluids:

- Paint (***shear thinning*** or ***pseudo-plastic***)
- Toothpaste (***Bingham plastic***)
- Quicksand (***shear thickening*** or ***dilatant***).

We consider only Newtonian fluids in this course.

For ***Newtonian fluids*** (see text for derivation), it turns out that

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu \dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & 2\mu \dfrac{\partial v}{\partial y} & \mu\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \mu\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & 2\mu \dfrac{\partial w}{\partial z} \end{pmatrix} \qquad \textbf{(9–57)}$$

We have achieved our goal of writing σ_{ij} in terms of pressure P, velocity components u, v, and w, and fluid viscosity μ.

Now we plug this expression for the stress tensor σ_{ij} into Cauchy's equation. The result is the famous ***Navier-Stokes equation***, shown here for incompressible flow.

Incompressible Navier–Stokes equation:

Navier-Stokes equation:

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V} \qquad \textbf{(9–60)}$$

To solve fluid flow problems, we need both the continuity equation and the Navier-Stokes equation. Since it is a vector equation, the Navier-Stokes equation is usually split into three components in order to solve fluid flow problems. In Cartesian coordinates,

Incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \textbf{(9–61a)}$$

x-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad \textbf{(9–61b)}$$

y-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \textbf{(9–61c)}$$

z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad \textbf{(9–61d)}$$