Derivation of Material Acceleration (Section 4-1)

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Acceleration of a fluid particle:

$$\vec{a}_{\text{particle}} = \frac{d\vec{V}_{\text{particle}}}{dt} \qquad \textbf{(4–6)}$$

This is a *Lagrangian* description of the acceleration of a fluid particle.

However, at any instant in time t, the velocity of the particle is the same as the local value of the velocity *field* at the location ($x_{\text{particle}}(t)$, $y_{\text{particle}}(t)$, $z_{\text{particle}}(t)$) of the particle, since the fluid particle moves with the fluid by definition. In other words, $\vec{V}_{\text{particle}}(t) \equiv \vec{V}(x_{\text{particle}}(t), y_{\text{particle}}(t), z_{\text{particle}}(t), t)$. To take the time derivative in Eq. 4–6, we must therefore use the *chain rule*, since the dependent variable ($\vec{V}$) is a function of *four* independent variables (x_{particle}, y_{particle}, z_{particle}, and t),

Recall the ***chain rule***: If f is a function of two variables, t and some variable s which is itself also a function of t, then we take the total derivative of f with respect to t as follows:

$$\frac{df}{dt} = \frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial s}\frac{ds}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial s}\frac{ds}{dt}$$

Now let's apply this chain rule to the time derivative of the fluid particle's velocity:

Note that from the Lagrangian description (following a fluid particle, x_{particle} is a function of time, since the particle's location changes with time. Thus, $x_{\text{particle}} = x_{\text{particle}}(t)$. Similarly, $y_{\text{particle}} = y_{\text{particle}}(t)$ and $z_{\text{particle}} = z_{\text{particle}}(t)$.

Thus, the acceleration of a fluid particle is calculated using the chain rule as follows:

$$\vec{a}_{\text{particle}} = \frac{d\vec{V}_{\text{particle}}}{dt} = \frac{d\vec{V}}{dt} = \frac{d\vec{V}(x_{\text{particle}}, y_{\text{particle}}, z_{\text{particle}}, t)}{dt}$$

$$= \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x_{\text{particle}}}\frac{dx_{\text{particle}}}{dt} + \frac{\partial \vec{V}}{\partial y_{\text{particle}}}\frac{dy_{\text{particle}}}{dt} + \frac{\partial \vec{V}}{\partial z_{\text{particle}}}\frac{dz_{\text{particle}}}{dt} \qquad \textbf{(4–7)}$$

$dt/dt =$

$dx_{\text{particle}}/dt =$

$dy_{\text{particle}}/dt =$

$dz_{\text{particle}}/dt =$

Or, finally,

$$\vec{a}_{\text{particle}}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} \qquad \textbf{(4–8)}$$