Nondimensionalization of the Navier-Stokes Equation

(Section 10-2, Çengel and Cimbala)

Nondimensionalization:

We begin with the differential equation for conservation of linear momentum for a Newtonian fluid, i.e., the ***Navier-Stokes equation***. For incompressible flow,

$$\rho \frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V} \qquad \textbf{(10–2)}$$

Equation 10-2 is *dimensional*, and each variable or property (ρ, $\vec{V}$, t, μ, etc.) is also *dimensional*. What are the primary dimensions (in terms of {m}, {L}, {t}, {T}, etc) of each term in this equation?

Answer: $\boxed{\{\quad\}}$

To nondimensionalize Eq. 10-2, we choose ***scaling parameters*** as follows:

TABLE 10–1

Scaling parameters used to nondimensionalize the continuity and momentum equations, along with their primary dimensions

Scaling Parameter	Description	Primary Dimensions
L	Characteristic length	{L}
V	Characteristic speed	$\{Lt^{-1}\}$
f	Characteristic frequency	$\{t^{-1}\}$
$P_0 - P_\infty$	Reference pressure difference	$\{mL^{-1}t^{-2}\}$
g	Gravitational acceleration	$\{Lt^{-2}\}$

We define ***nondimensional variables***, using the scaling parameters in Table 10-1:

$$t^* = ft \qquad \vec{x}^* = \frac{\vec{x}}{L} \qquad \vec{V}^* = \frac{\vec{V}}{V}$$

$$P^* = \frac{P - P_\infty}{P_0 - P_\infty} \qquad \vec{g}^* = \frac{\vec{g}}{g} \qquad \vec{\nabla}^* = L\vec{\nabla} \qquad \textbf{(10–3)}$$

To plug Eqs. 10-3 into Eq. 10-2, we need to first rearrange the equations in terms of the dimensional variables, i.e.,

$$t = \frac{1}{f}t^* \qquad \vec{x} = L\vec{x}^* \qquad \vec{V} = V\vec{V}^*$$

$$P = P_\infty + \left(P_0 - P_\infty\right)P^* \qquad \vec{g} = g\vec{g}^* \qquad \vec{\nabla} = \frac{1}{L}\vec{\nabla}^*$$

Now we substitute all of the above into Eq. 10-2 to obtain

$$\rho V f \frac{\partial \vec{V}^*}{\partial t^*} + \frac{\rho V^2}{L}\left(\vec{V}^* \cdot \vec{\nabla}^*\right)\vec{V}^* = -\frac{P_0 - P_\infty}{L}\vec{\nabla}^* P^* + \rho g \vec{g}^* + \frac{\mu V}{L^2}\nabla^{*2}\vec{V}^*$$

Every additive term in the above equation has primary dimensions $\{m^1L^{-2}t^{-2}\}$. To nondimensionalize the equation, we multiply every term by constant $L/(\rho V^2)$, which has primary dimensions $\{m^{-1}L^2t^2\}$, so that the dimensions cancel. After some rearrangement,

$$\boxed{\left[\frac{fL}{V}\right]}\frac{\partial \vec{V}^*}{\partial t^*} + \left(\vec{V}^* \cdot \vec{\nabla}^*\right)\vec{V}^* = -\boxed{\left[\frac{P_0 - P_\infty}{\rho V^2}\right]}\vec{\nabla}^* P^* + \boxed{\left[\frac{gL}{V^2}\right]}\vec{g}^* + \boxed{\left[\frac{\mu}{\rho VL}\right]}\nabla^{*2}\vec{V}^* \qquad \textbf{(10–5)}$$

Strouhal number, where $\mathrm{St} = \frac{fL}{V}$

Euler number, where $\mathrm{Eu} = \frac{P_0 - P_\infty}{\rho V^2}$

Inverse of Froude number squared, where $\mathrm{Fr} = \frac{V}{\sqrt{gL}}$

Inverse of Reynolds number, where $\mathrm{Re} = \frac{\rho VL}{\mu}$

Thus, Eq. 10-5 can therefore be written as

Navier-Stokes equation in nondimensional form:

$$[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^* \cdot \vec{\nabla}^*)\vec{V}^* = -[\mathrm{Eu}]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^* \qquad \textbf{(10–6)}$$

Nondimensionalization **vs.** ***Normalization***:

Equation 10-6 above is *nondimensional*, but not necessarily *normalized*. What is the difference?

- ***Nondimensionalization*** concerns only the *dimensions* of the equation – we can use *any* value of scaling parameters *L*, *V*, etc., and we always end up with Eq. 10-6.
- ***Normalization*** is more restrictive than nondimensionalization. To *normalize* the equation, we must choose scaling parameters *L*, *V*, etc. that are appropriate for the flow being analyzed, such that ***all nondimensional variables*** (t^*, $\vec{V}^*$, P^*, etc.) ***in Eq. 10-6 are of order of magnitude unity***. In other words, their minimum and maximum values are reasonably close to 1.0 (e.g., $-6 < P^* < 3$, or $0 < P^* < 11$, but *not* $0 < P^* < 0.001$, or $-200 < P^* < 500$). We express the normalization as follows:

$$t^* \sim 1, \quad \vec{x}^* \sim 1, \quad \vec{V}^* \sim 1, \quad P^* \sim 1, \quad \vec{g}^* \sim 1, \quad \vec{\nabla}^* \sim 1$$

If we have properly normalized the Navier-Stokes equation, we can compare the relative importance of various terms in the equation by comparing the *relative magnitudes* of the nondimensional parameters St, Eu, Fr, and Re.