Comprehensive Equation Sheet for M E 320

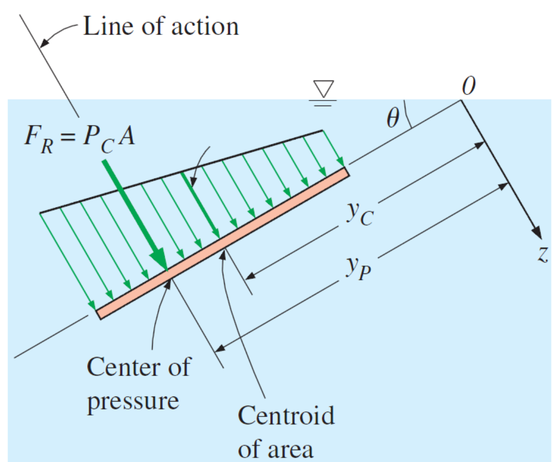
Author: John M. Cimbala, Penn State University. Latest revision: 03 February 2015

Notation for this equation sheet: $\mathcal{V}$ = volume, V = velocity, v = y-component of velocity, ν = kinematic viscosity

- **Constants**: $g = 9.807 \frac{\text{m}}{\text{s}^2}$ $P_{\text{atm}} = 101.3 \text{ kPa}$ $\rho_{\text{air}} = 1.204 \frac{\text{kg}}{\text{m}^3}$ $\rho_{\text{water}} = 998.0 \frac{\text{kg}}{\text{m}^3}$ $\rho_{\text{mercury}} \cong 13,600 \frac{\text{kg}}{\text{m}^3}$
- **Conversions**: $\left(\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\right)$ $\left(\frac{\text{Pa}\cdot\text{m}^2}{\text{N}}\right)$ $\left(\frac{\text{kW}\cdot\text{s}}{1000\text{ N}\cdot\text{m}}\right)$ $\left(\frac{\text{m}^3}{1000\text{ L}}\right)$ $\left(\frac{2\pi\text{ rad}}{\text{rotation}}\right)$ $T(\text{K}) \cong T(^\circ\text{C}) + 273.15$
- **Specific gravity**: $SG = \rho / \rho_{H_2O}$ **Ideal gas**: $P = \rho RT$ $R = c_P - c_v$ $k = c_P / c_v$ $k_{\text{air}} = 1.40$ $R_{\text{air}} \cong 300 \frac{\text{m}^2}{\text{s}^2\cdot\text{K}}$
- **Simple shear flow and viscosity**: $u(y) = Vy/h$ for flow sandwiched between two infinite flat plates; $\tau = \mu \frac{du}{dy}$
- **Surface tension**: $\Delta P_{\text{droplet}} = P_{\text{inside}} - P_{\text{outside}} = 2\frac{\sigma_s}{R}$ $\Delta P_{\text{bubble}} = P_{\text{inside}} - P_{\text{outside}} = 4\frac{\sigma_s}{R}$ **Capillary tube**: $h = \frac{2\sigma_s}{\rho g R}\cos\phi$
- **Gage, vacuum, and vapor pressure**: $P_{\text{gage}} = P_{\text{absolute}} - P_{\text{atm}}$ $P_{\text{vacuum}} = P_{\text{atm}} - P_{\text{absolute}}$ $P_v \cong 2.0$ kPa for water at $T \cong 20^\circ\text{C}$
- **Hydrostatics**: $\vec{\nabla}P = \rho\vec{g}$ $\frac{dP}{dz} = -\rho g$ $P_{\text{below}} = P_{\text{above}} + \rho g|\Delta z|$ $\Delta P = \rho g h$
- **Buoyant force**: $F_B = \rho_f g \mathcal{V}_{\text{submerged}}$
- **Forces on submerged, plane surfaces (see sketch)**:

  $F = (P_0 + \rho g h_C)A = P_C A = P_{\text{avg}} A$ $y_P = y_C + \frac{I_{xx,C}}{\left[y_C + P_0/(\rho g \sin\theta)\right]A}$

  Rectangle a wide and b tall: $I_{xx,C} = \frac{ab^3}{12}$. Circle of radius R: $I_{xx,C} = \frac{\pi R^4}{4}$



- **Rigid body acceleration**: $\vec{\nabla}P = \rho\vec{G} = \rho(\vec{g} - \vec{a})$. Use modified gravity vector $\vec{G}$ in place of $\vec{g}$ everywhere.
- **Rigid body rotation (vertical axis, cylinder radius *R*)**:

  $z_{\text{surface}} = h_0 - \frac{\omega^2}{4g}\left(R^2 - 2r^2\right)$ $P = P_0 + \frac{\rho\omega^2}{2}r^2 - \rho g z$
- **Acceleration field and material derivative**: $\vec{a}(x,y,z,t) = \frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}$ $\frac{Db}{Dt} = \frac{\partial b}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)b$
- **Vorticity vector**: $\vec{\zeta} = \vec{\nabla}\times\vec{V} = \text{curl}\left(\vec{V}\right) = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$
- **Equation for a 2-D streamline**: $\left.\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u}$ where in Cartesian coordinates, $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$
- **Rates of motion and deformation of fluid particles:** $\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$ $\varepsilon_{xx} = \frac{\partial u}{\partial x},\ \varepsilon_{yy} = \frac{\partial v}{\partial y},\ \varepsilon_{zz} = \frac{\partial w}{\partial z}$

  $\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$ $\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),\ \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$
- **Volumetric strain rate in Cartesian coordinates**: $\frac{1}{\mathcal{V}}\frac{D\mathcal{V}}{Dt} = \frac{1}{\mathcal{V}}\frac{d\mathcal{V}}{dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

- **Strain rate tensor in Cartesian coordinates**: $$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

- **Reynolds transport theorem (RTT)**: For a fixed or non-fixed control volume, where B = some flow property, and $b = B/m$,

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \rho b dV + \int_{\text{CS}} \rho b \vec{V}_{\text{r}} \cdot \vec{n} dA$$ where $\vec{V}_{\text{r}} = \vec{V} - \vec{V}_{\text{CS}}$ is the relative velocity

- **Conservation of mass for a CV**: $\frac{dm_{\text{CV}}}{dt} = \sum_{\text{in}} \dot{m} - \sum_{\text{out}} \dot{m}$ $\dot{m} = \rho \dot{V} = \rho V_{\text{avg}} A_c$ at an inlet or outlet, where $V_{\text{avg}} = \frac{1}{A_c}\int V_{\text{normal}} dA_c$

For ***steady-state, steady flow*** (***SSSF***) problems: $\sum_{\text{in}} \dot{m} = \sum_{\text{out}} \dot{m}$. For ***incompressible SSSF***: $\sum_{\text{in}} \dot{V} = \sum_{\text{out}} \dot{V}$.

- **Conservation of energy for a CV**: $$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \frac{d}{dt}\int_{\text{CV}} e\rho dV + \sum_{\text{out}} \dot{m}\left(h + \alpha\frac{V^2}{2} + gz\right)_{\text{avg}} - \sum_{\text{in}} \dot{m}\left(h + \alpha\frac{V^2}{2} + gz\right)_{\text{avg}}$$

where $e = u + \frac{V^2}{2} + gz$ $h = u + \frac{P}{\rho}$ $\dot{m} = \rho V_{\text{avg}} A_c$ $\alpha = \frac{1}{A_c}\int_{A_c}\left(\frac{V}{V_{\text{avg}}}\right)^3 dA_c$ $\alpha = 2.0$ fully developed laminar pipe flow; $\alpha \approx 1.05$ fully developed turbulent pipe flow

- **Bernoulli equation**: $\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2$ along a streamline from 1 to 2.

- **Momentum equation for a CV**: $$\sum \vec{F} = \sum \vec{F}_{\text{gravity}} + \sum \vec{F}_{\text{pressure}} + \sum \vec{F}_{\text{viscous}} + \sum \vec{F}_{\text{other}} = \frac{d}{dt}\int_{CV} \rho \vec{V} dV + \sum_{\text{out}} \beta \dot{m} \vec{V} - \sum_{\text{in}} \beta \dot{m} \vec{V}$$

- **Head form of energy equation**: $\frac{P_1}{\rho_1 g} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_{\text{pump, }u} = \frac{P_2}{\rho_2 g} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_{\text{turbine, }e} + h_L$, where 1 = inlet, 2 = outlet, and the useful pump head and extracted turbine head are $h_{\text{pump, }u} = \frac{\eta_{\text{pump}} \dot{W}_{\text{pump shaft}}}{\dot{m} g}$ and $h_{\text{turbine, }e} = \frac{\dot{W}_{\text{turbine shaft}}}{\eta_{\text{turbine}} \dot{m} g}$

- **Turbomachinery**: ***Net head*** = $H = h_{\text{pump,u}} = \left(\frac{P}{\rho g} + \frac{V^2}{2g} + z\right)_{\text{out}} - \left(\frac{P}{\rho g} + \frac{V^2}{2g} + z\right)_{\text{in}}$

Water horsepower = $\dot{W}_{\text{pump,u}} = \dot{m} g H = \rho \dot{V} g H$, ***Brake horsepower*** = $\dot{W}_{\text{shaft}} = \text{bhp}$

To match a pump (available head) to a piping system (required head), match $H_{\text{available}} = H_{\text{required}}$

Pump and turbine performance parameters:

$C_Q = \text{Capacity coefficient} = \frac{\dot{V}}{\omega D^3}$ $C_H = \text{Head coefficient} = \frac{gH}{\omega^2 D^2}$ $C_P = \text{Power coefficient} = \frac{\text{bhp}}{\rho \omega^3 D^5}$

$\eta_{\text{pump}} = \frac{\dot{W}_{\text{pump,u}}}{\text{bhp}} = \frac{\rho \dot{V} g H}{\text{bhp}} = \frac{C_Q C_H}{C_P} = \text{function of } C_Q$ $\eta_{\text{turbine}} = \frac{\text{bhp}}{\dot{W}_{\text{turbine,e}}} = \frac{\text{bhp}}{\rho \dot{V} g H} = \frac{C_P}{C_Q C_H} = \text{function of } C_P$

Pump and turbine affinity laws: $\frac{\dot{V}_{\text{B}}}{\dot{V}_{\text{A}}} = \frac{\omega_{\text{B}}}{\omega_{\text{A}}}\left(\frac{D_{\text{B}}}{D_{\text{A}}}\right)^3$ $\frac{H_{\text{B}}}{H_{\text{A}}} = \left(\frac{\omega_{\text{B}}}{\omega_{\text{A}}}\right)^2\left(\frac{D_{\text{B}}}{D_{\text{A}}}\right)^2$ $\frac{\text{bhp}_{\text{B}}}{\text{bhp}_{\text{A}}} = \frac{\rho_{\text{B}}}{\rho_{\text{A}}}\left(\frac{\omega_{\text{B}}}{\omega_{\text{A}}}\right)^3\left(\frac{D_{\text{B}}}{D_{\text{A}}}\right)^5$

- **Grade lines**: ***Energy Grade Line*** = $\text{EGL} = \frac{P}{\rho g} + \frac{V^2}{2g} + z$, ***Hydraulic Grade Line*** = $\text{HGL} = \frac{P}{\rho g} + z$

- **Pipe flows**: $\text{Re} = \frac{\rho V D}{\mu} = \frac{VD}{\nu}$ $h_{L,\text{ total}} = \sum h_{L,\text{ major}} + \sum h_{L,\text{ minor}}$, ***Major head loss*** = $h_L = f\frac{L}{D}\frac{V^2}{2g}$ $f = \frac{8\tau_w}{\rho V^2} = \text{fnc}\left(\text{Re}, \frac{\varepsilon}{D}\right)$,

  $\dot{m} = \rho V A$, ***Minor head loss*** = $h_L = K_L \frac{V^2}{2g}$, ***Fully developed laminar*** pipe flow, $\alpha = 2$ $f = 64/\text{Re}$. Near wall, $\tau_\text{wall} = \mu \left.\frac{du}{dy}\right)_\text{wall}$

  Fully developed turbulent pipe flow, $\alpha \cong 1.05$ $\frac{1}{\sqrt{f}} \approx -2.0\log_{10}\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{\text{Re}\sqrt{f}}\right)$ (Colebrook eq., Moody chart)

- **Hydraulic diameter**: $D_h = \frac{4A_c}{p}$, where A_c is the cross-sectional area and p is the *wetted* perimeter

- **Continuity equation**: $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho \vec{V}\right) = 0$. If *incompressible*, $\vec{\nabla}\cdot\vec{V} = 0$

  In ***Cartesian*** coordinates, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, and in ***cylindrical*** coordinates, $\frac{1}{r}\frac{\partial(r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$

- **Stream function**: Cartesian (x-y plane): $u = \frac{\partial \psi}{\partial y}$ $v = -\frac{\partial \psi}{\partial x}$; Cylindrical planar ($r$-$\theta$ plane): $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ $u_\theta = -\frac{\partial \psi}{\partial r}$

- **Navier-Stokes equation**: (for incompressible, Newtonian flow) $\rho\frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}$

- **Creeping flow**: For Re << 1, $\vec{\nabla}P \approx \mu\nabla^2\vec{V}$.

- **Potential flow (Irrotational flow)**: Since $\vec{\nabla}\times\vec{V} = 0$, then $\vec{V} = \vec{\nabla}\phi$, where ϕ is the velocity potential function. $\nabla^2\phi = 0$

  Cartesian coordinates: $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$. Cylindrical coordinates: $\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$

  If flow is also 2-D, then $\nabla^2\psi = 0$ as well. Superposition of both ϕ and ψ is valid for potential flow.

- **Boundary layers**: $\text{Re}_x = \frac{\rho U x}{\mu} = \frac{Ux}{\nu}$, where x is *along* the body. $U(x)$ is the outer flow (just outside the boundary layer).

  For steady flow, continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$; x-momentum: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$; y-momentum: $\frac{\partial P}{\partial y} \approx 0$

- **Flat plate boundary layer**:

  If ***laminar***, ($\text{Re}_x < 5\times10^5$), $\frac{\delta}{x} = \frac{4.91}{\sqrt{\text{Re}_x}}$ $\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\text{Re}_x}}$ $C_{f,x} = \frac{2\tau_w}{\rho U^2} = \frac{0.664}{\sqrt{\text{Re}_x}}$ $C_f = C_D = \frac{1.33}{\sqrt{\text{Re}_x}}$

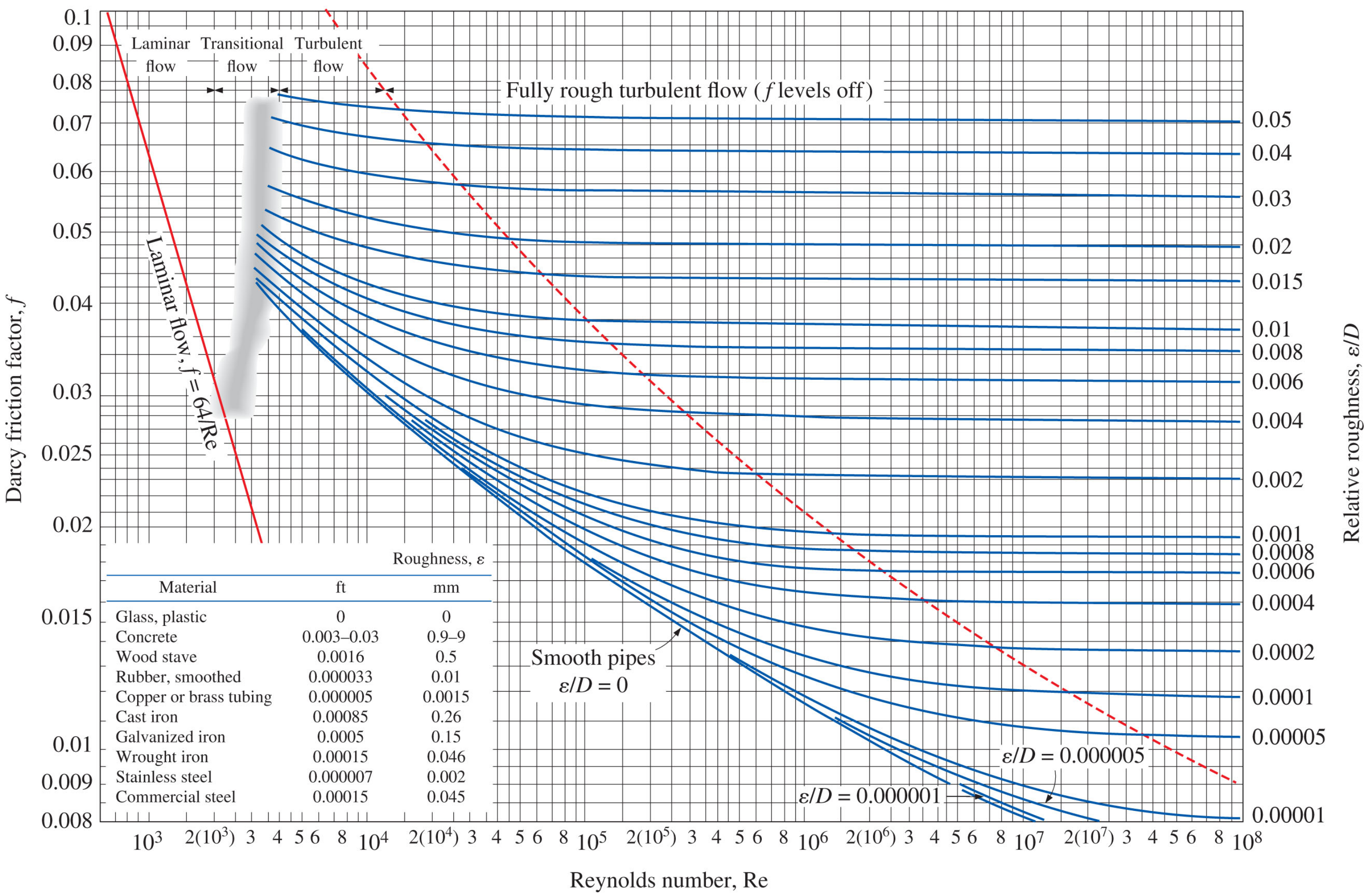
  If ***turbulent*** and ***smooth***, ($5\times10^5 < \text{Re}_x < 10^7$), $\frac{\delta}{x} \approx \frac{0.38}{(\text{Re}_x)^{1/5}}$ $\frac{\delta^*}{x} \approx \frac{0.048}{(\text{Re}_x)^{1/5}}$ $C_{f,x} \approx \frac{0.059}{(\text{Re}_x)^{1/5}}$ $C_f = C_D = \frac{0.074}{\text{Re}_x^{1/5}}$

- **Drag and Lift on bodies**: $C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A}$ $C_L = \frac{F_L}{\frac{1}{2}\rho V^2 A}$, where A = projected frontal area *or* planform area. C_D includes skin friction *and* pressure drag. For bodies without ground effect, required power = $\dot{W} = F_D V$. For vehicles in ground effect, the required ***power to the wheels*** = $\dot{W} = \mu_\text{rolling} W V + \frac{1}{2}\rho V^3 C_D A$, where μ_rolling = coefficient of rolling resistance, and W is the vehicle weight.

- **Isentropic compressible flow for air (k = 1.4)**: $\frac{T_0}{T} = 1 + 0.2\text{Ma}^2$ $\frac{\rho_0}{\rho} = \left(1 + 0.2\text{Ma}^2\right)^{2.5}$ $\frac{P_0}{P} = \left(1 + 0.2\text{Ma}^2\right)^{3.5}$ $\dot{m}_\text{max} = \frac{0.6847 P_0 A^*}{\sqrt{RT_0}}$

- **Normal shock equations for air (k = 1.4)**: $\frac{T_2}{T_1} = \left(2 + 0.4\text{Ma}_1^2\right)\frac{2.8\text{Ma}_1^2 - 0.4}{5.76\text{Ma}_1^2}$ $\frac{P_2}{P_1} = \frac{\left(2.8\text{Ma}_1^2 - 0.4\right)}{2.4}$ $\frac{\rho_2}{\rho_1} = \frac{2.4\text{Ma}_1^2}{\left(2 + 0.4\text{Ma}_1^2\right)}$

- **Moody Chart**:



Material	Roughness, ε ft	Roughness, ε mm
Glass, plastic	0	0
Concrete	0.003–0.03	0.9–9
Wood stave	0.0016	0.5
Rubber, smoothed	0.000033	0.01
Copper or brass tubing	0.000005	0.0015
Cast iron	0.00085	0.26
Galvanized iron	0.0005	0.15
Wrought iron	0.00015	0.046
Stainless steel	0.000007	0.002
Commercial steel	0.00015	0.045