

Windowing with FFTs

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Introduction

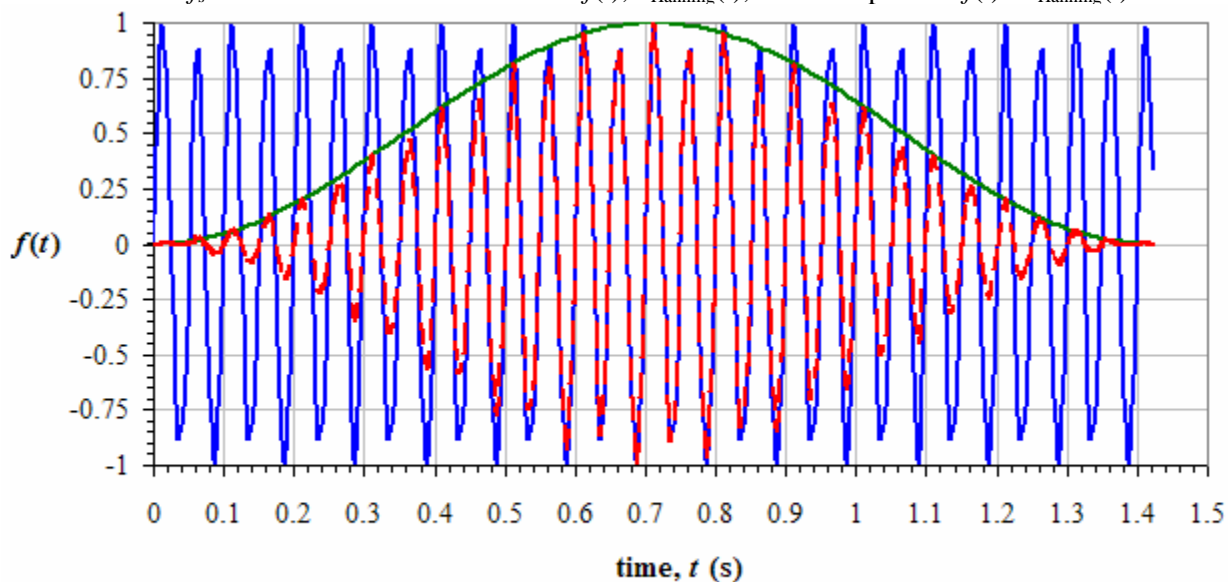
- When performing FFTs, the sampled time signal does not always start and end at exactly the same phase of the signal. This leads to an error in the frequency spectrum called *leakage*, in which peaks in the spectrum are spread out, and are therefore not as sharply defined as they should be.
- The goal of *windowing* is to **reduce the effect of leakage**; we discuss windowing in this learning module.

Windowing for signals with no DC offset

- Can leakage be reduced without increasing the number of sampled data points N or the sampling time T ? The answer is *yes*, through a process called *windowing*.
- As mentioned previously, **the cause of leakage is that the signal starts and ends at different phases in the cycle**. In most experiments, the sampling rate and number of data points is such that this cannot be avoided. However, if the amplitude of the signal at both ends (near $t = 0$ and near $t = T$) is reduced to zero in a smooth fashion, the end effects will not be as serious. This is the fundamental concept of windowing.
- In practice, the signal is multiplied by a *windowing function*, which starts at zero at $t = 0$, increases smoothly to 1 at $t = T/2$, and then decreases smoothly again to zero at $t = T$.
- Many such windowing functions have been invented, but the most common one is the *Hanning window*, created by a simple cosine function. The *Hanning window function*, $u_{\text{Hanning}}(t)$ is defined as

$$u_{\text{Hanning}}(t) = \frac{1}{2} \left[1 - \cos\left(\frac{2\pi t}{T}\right) \right] \text{ for the entire range of the time trace, } 0 \leq t \leq T.$$

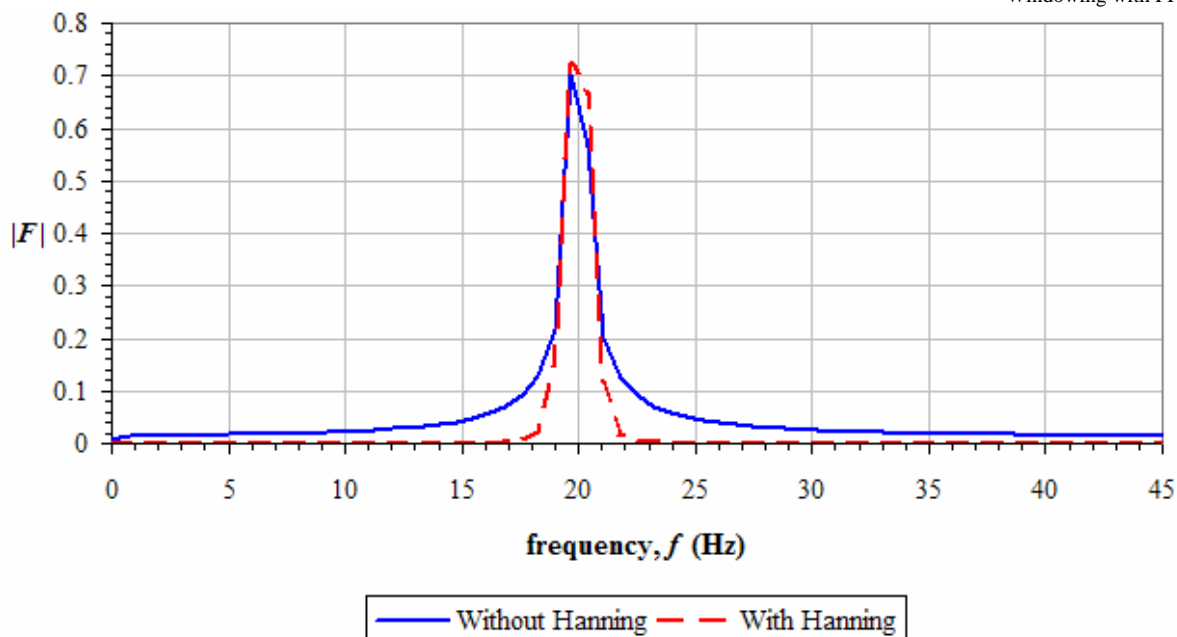
- As an example, consider the input signal to be a 20 Hz sine wave function $f(t)$ of amplitude 1.0 V, and no DC offset or phase shift. For this example, the sampling frequency is $f_s = 90$ Hz, and $N = 128$. The total sampling time is thus $T = N/f_s = 1.4222\dots$ s. Plotted below are $f(t)$, $u_{\text{Hanning}}(t)$, and their product $f(t) \cdot u_{\text{Hanning}}(t)$:



— perceived signal — Hanning function — signal * Hanning

Notice that the modified time trace $f(t) \cdot u_{\text{Hanning}}(t)$ trails off to zero at both ends ($t = 0$ and $t = T$). The middle region of the time trace is relatively unaffected by the windowing function. In essence, **the windowing operation emphasizes the middle portion of the time trace, and de-emphasizes the ends**.

- Does this reduce leakage? Yes, as can be seen in the frequency spectra below. Note that since the sampling frequency is 90 Hz in both cases, the maximum frequency or folding frequency is half of this, or 45 Hz. Shown below is the frequency spectrum of the original signal, $f(t)$, and on the same plot the frequency spectrum of the modified (windowed) signal, $f(t) \cdot u_{\text{Hanning}}(t)$. The modified FFT clearly has less leakage, although leakage still exists to some degree.



- The Hanning windowing operation causes a loss of information near the end points of the signal. This leads to an overall reduction in the amplitude of the frequency spectrum. To compensate, it turns out that the amplitude of the FFT must be multiplied by a factor of $\sqrt{8/3}$, which brings the amplitude back up to the proper level.
- The factor of $\sqrt{8/3}$ has been included in the above plot. As can be seen on the plot, the peak amplitude of the modified spectrum is approximately the same as that of the original spectrum (it has improved ever so slightly), but leakage has indeed been reduced significantly – the peak is much narrower.
- As explained above, this is *still* lower than the known input amplitude of 1.0 volts because of leakage.
- The bottom line: **Windowing reduces leakage, but does not totally eliminate leakage.** [If leakage had been totally eliminated, the frequency spectrum would show a single infinitesimally thin spike at 20 Hz with an amplitude of exactly 1.0 V.]

Windowing for signals *with* a DC offset

- We need to be more careful when applying windowing to an input signal that has a DC offset.
- Simple application of the windowing function, as described above, would introduce two problems, since the modified (windowed) input signal is forced to be zero at the beginning and end of the sampling time:
 - A false DC offset at $f = 0$ (approximately half of the true DC offset).
 - A false amplitude at $f = \Delta f$ (the frequency of the window function itself).
- **These problems will be demonstrated and discussed in more detail in class.**
- To properly apply windowing to an input signal with a DC offset, we follow this procedure:
 - Calculate the mean value of the input data.
 - Subtract the mean value from each data point. [This effectively removes or *uncouples* the DC offset – forcing the data to have zero mean.]
 - Multiply the modified (uncoupled) input data by the windowing function as discussed previously, and perform the FFT on the windowed signal.
 - Manipulate each value of the FFT output as usual (calculate the modulus, divide by $N/2$, and multiply by $\sqrt{8/3}$ for the Hanning window). Also remember to divide the *first* amplitude (at $f = 0$) by an additional factor of 2.
 - Lastly, add the DC offset to the first amplitude (at $f = 0$), and plot the frequency spectrum.
- When this procedure is followed, the correct DC offset is preserved, and leakage is reduced. Specifically, any peaks in the spectrum are narrowed as illustrated previously.
- *Final comment:* When the beginning and ending of the sampled data are at different phases of the periodic input signal, the last portion of the sampled data contains a fraction of a complete period. Therefore, the mean value and the DC offset calculated using the above method will *not* be exact, but will be *approximate*. This error can be reduced (but never fully eliminated) by sampling a larger number of data points.