Experimental Uncertainty Analysis

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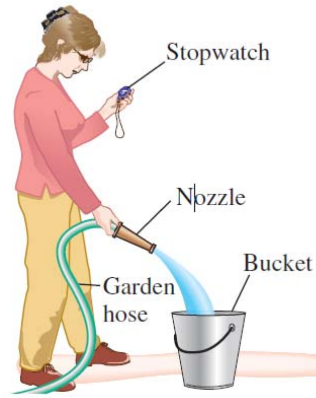
- Now that the principles of measurement uncertainty and confidence level have been established, we can predict the uncertainty of a *calculated* quantity as well.
- ***Experimental uncertainty analysis*** provides a method for ***predicting the uncertainty of a variable based on its component uncertainties.***
- Some authors call this analysis the ***propagation of uncertainty***.
- Suppose we measure N physical quantities (or variables, like voltage, resistance, power, torque, temperature, etc.), $x_1, x_2, ..., x_N$. Also suppose that each of these quantities has a known ***experimental uncertainty*** associated with it, which we shall denote by u_{x_i}, i.e., $\boxed{x_i = \bar{x}_i \pm u_{x_i}}$.
- Furthermore, unless otherwise specified, each of these uncertainties has a confidence level of 95%. Since the x_i variables are components of the calculated quantity, we call the uncertainties ***component uncertainties***.
- Suppose now that some new variable, R, is a function of these measured quantities, i.e., $\boxed{R = R(x_1, x_2, ..., x_N)}$. The goal in experimental uncertainty analysis is to estimate the uncertainty in R to the same confidence level as that of the component uncertainties, i.e., we want to report R as $\boxed{R = \bar{R} \pm u_R}$, where u_R is the ***predicted uncertainty*** on variable R. There are two types of uncertainty on variable R:
  - ***Maximum uncertainty*** – We define the ***maximum uncertainty*** on variable R as $\boxed{u_{R,\max} = \sum_{i=1}^{i=N} \left| u_{x_i} \frac{\partial R}{\partial x_i} \right|}$.
    - Because of the absolute value signs, this expression assumes that *all* the errors in the component variable x_i measurements are such that the error in R is always the *same sign*.
    - Such a case would be highly unlikely, especially for a large number of variables (large N), because some of the errors would be positive and some negative, and the errors would cancel each other out somewhat. In other words, this is a ***worst case scenario***.
  - ***Expected uncertainty*** – We define the ***expected uncertainty*** on variable R as $\boxed{u_{R,\mathrm{RSS}} = \sqrt{\sum_{i=1}^{i=N} \left(u_{x_i} \frac{\partial R}{\partial x_i} \right)^2}}$.
    - Expected uncertainty is also called the ***root of the sum of the squares uncertainty***, or ***RSS uncertainty***, because of the above equation – the square root of the sum of squared quantities.
    - This uncertainty estimate is more realistic than the maximum uncertainty since it is unlikely that the maximum error will occur on all component variables simultaneously.
    - RSS uncertainty is the ***engineering standard***, and the usual notation is to set u_R equal to the RSS uncertainty, i.e., $\boxed{u_R = u_{R,\mathrm{RSS}}}$.
    - It turns out that we can write $\boxed{R = (\text{calculated } R) \pm u_R}$, to ***the same confidence level*** as that of each of the x_i measurements (the same confidence level as that of the individual component measurements).
    - It is useful to define the ***relative RSS uncertainty*** as $\boxed{u_R / R}$. Since u_R and R have the same dimensions and units, the relative RSS uncertainty is always a dimensionless value.
- We use the RSS uncertainty as the standard for experimental uncertainty analysis in this course.
- A simpler formula for RSS results if the functional form of $R = R(x_1, x_2, ..., x_N)$ contains only *multiplications* or *divisions* of the component variables. In general, if R is of the form $\boxed{R = C \cdot x_1^{a_1} \cdot x_2^{a_2} \ldots \cdot x_N^{a_N}}$, where C is a constant, and $a_1, a_2, \ldots a_N$ are the exponents of each measured variable $x_1, x_2, \ldots x_N$, It can be shown that the relative RSS uncertainty of R takes the simpler form $\boxed{\frac{u_R}{R} = \frac{u_{R,\mathrm{RSS}}}{R} = \sqrt{\sum_{i=1}^{i=N} \left(a_i \frac{u_{x_i}}{x_i} \right)^2}}$. The really nice thing about this expression is that it is ***dimensionless***, since all uncertainties are *relative*.
- The above simpler expression for relative RSS uncertainty can easily be proven by working out the derivatives of the more general equation for $u_{R,\mathrm{RSS}}$ given earlier.
- ***Caution***: This simpler expression is valid only for cases in which R is of the form $R = C \cdot x_1^{a_1} \cdot x_2^{a_2} \ldots \cdot x_N^{a_N}$. Do *not* use this simpler expression for more complicated functions, e.g., with trig functions, $\exp(x_i)$, etc.

Example:

Given: The volume flow rate $\dot{V}$ through a garden hose is calculated (not measured directly) as a function of measured volume V and measured time t. Experimentally, this is accomplished by measuring the time it takes to fill up a container of known volume. The equation for volume flow rate is $\dot{V} = V/t$. The mean values and experimental uncertainties in measuring V and t are known: $V = 1.15 \pm 0.05$ gal, and $t = 33.0 \pm 0.1$ s, with 95% confidence.



To do: Calculate $\dot{V}$ and its estimated uncertainties (maximum and RSS) in gallons per minute (gpm).

Solution:

- First we calculate the mean value: $\dot{V} = \dfrac{V}{t} = \dfrac{1.15 \text{ gal}}{33.0 \text{ s}}\left(\dfrac{60 \text{ s}}{\text{min}}\right) = 2.0909\dfrac{\text{gal}}{\text{min}}$.

  To three significant digits, we express the result as $\boxed{\dot{V} = 2.09\dfrac{\text{gal}}{\text{min}}}$.

- From the equation $\dot{V} = \dfrac{V}{t}$, we calculate the partial derivatives: $\dfrac{\partial \dot{V}}{\partial V} = \dfrac{1}{t}$ and $\dfrac{\partial \dot{V}}{\partial t} = -\dfrac{V}{t^2}$.

- We let $R = \dot{V}$ for convenience. We know that $u_t = 0.1$ s and $u_V = 0.05$ gal. The ***maximum uncertainty*** is then calculated as $u_{R,\max} = \sum_{i=1}^{i=N}\left|u_{x_i}\dfrac{\partial R}{\partial x_i}\right| = \left|u_V\dfrac{\partial \dot{V}}{\partial V}\right| + \left|u_t\dfrac{\partial \dot{V}}{\partial t}\right| = \left|u_V\dfrac{1}{t}\right| + \left|u_t\left(-\dfrac{V}{t^2}\right)\right| = u_V\dfrac{1}{t} + u_t\dfrac{V}{t^2}$. Substitution of the values yields $u_{R,\max} = (0.05 \text{ gal})\dfrac{1}{33.0 \text{ s}} + (0.1 \text{ s})\dfrac{1.15 \text{ gal}}{(33.0 \text{ s})^2} = 0.00162\dfrac{\text{gal}}{\text{s}}\left(\dfrac{60 \text{ s}}{\text{min}}\right) = 0.0972\dfrac{\text{gal}}{\text{min}}$.

  Thus, the predicted maximum uncertainty in volume flow rate is $\boxed{u_{R,\max} = 0.0972\dfrac{\text{gal}}{\text{min}}}$.

- Next, we calculate the ***expected uncertainty*** (RSS uncertainty): $u_{R,\text{RSS}} = \sqrt{\sum_{i=1}^{i=N}\left(u_{x_i}\dfrac{\partial R}{\partial x_i}\right)^2} =$

  $\sqrt{\left(u_V\dfrac{\partial \dot{V}}{\partial V}\right)^2 + \left(u_t\dfrac{\partial \dot{V}}{\partial t}\right)^2} = \sqrt{\left(u_V\dfrac{1}{t}\right)^2 + \left(u_t\dfrac{-V}{t^2}\right)^2} = \sqrt{\left((0.05 \text{ gal})\dfrac{1}{33.0 \text{ s}}\right)^2 + \left((0.1 \text{ s})\dfrac{-1.15 \text{ gal}}{(33.0 \text{ s})^2}\right)^2} =$

  $0.00152\dfrac{\text{gal}}{\text{s}}\left(\dfrac{60 \text{ s}}{\text{min}}\right) = 0.0911\dfrac{\text{gal}}{\text{min}}$. Thus, the predicted RSS uncertainty is $\boxed{u_R = u_{R,\text{RSS}} = 0.0911\dfrac{\text{gal}}{\text{min}}}$.

- Alternately, since the equation $\dot{V} = V / t$ contains only multiplications and divisions, we can calculate the RSS uncertainty using the simpler formula:
  - We re-write the equation as $\dot{V} = V^1 t^{-1}$ so that the exponents are $a_V = 1$ and $a_t = -1$, for variables V and t, respectively.
  - The simpler equation is thus $\dfrac{u_R}{R} = \dfrac{u_{R,\text{RSS}}}{R} = \sqrt{\sum_{i=1}^{i=N}\left(a_i\dfrac{u_{x_i}}{x_i}\right)^2} = \sqrt{\left(a_V\dfrac{u_V}{V}\right)^2 + \left(a_t\dfrac{u_t}{t}\right)^2} =$

    $\sqrt{\left((1)\dfrac{0.05 \text{ gal}}{1.15 \text{ gal}}\right)^2 + \left((-1)\dfrac{0.1 \text{ s}}{33.0 \text{ s}}\right)^2} = 0.04358$. Thus, $u_R = \dfrac{u_R}{R}R = (0.04358)\left(2.0909\dfrac{\text{gal}}{\text{min}}\right) =$

    $0.0911\dfrac{\text{gal}}{\text{min}}$. This result agrees with the previous RSS uncertainty, as it must.

- Finally, the proper way to write the answer to the question "What is $\dot{V}$ in ***standard engineering format***?" is thus $\boxed{\dot{V} = 2.09 \pm 0.091\dfrac{\text{gal}}{\text{min}}}$ (with 95% confidence).

Discussion: Notice that the units of the uncertainties (the u terms) agree with those of the variables themselves ($\dot{V}$ – which we call R here, V, and t), as they must. Thus, unit issues are not important until the end when some unit conversions might be required, as here. As an aside comment, this illustrates the

principle of dimensional homogeneity, i.e., all additive terms in an equation must have the same dimensions and the same units.

- Some more comments about this example problem:
  - The meaning of the result is that the probability that the value of $\dot{V}$ lies within the indicated uncertainty (+/- 0.091 gpm) is 95%. In other words, we can be 95% confident in the result.
  - Notice that $\dot{V}$ is written to only three significant digits. Any more than that would be misleading since both volume and time are measured to only three significant digits. It would *not* be proper to write the answer as $\dot{V}$ = 2.0909091 +/- 0.091129626 gpm, even though that's what the calculator displays!
  - In this example, the volume measurement is the most significant source of error, and dominates the calculation of uncertainty. Nondimensionally, the importance of each of the two contributors to the uncertainty is indicated by the ***relative uncertainty*** (also called the ***fractional uncertainty***), which is simply the uncertainty divided by the value for each variable. Here, the relative uncertainties are $\frac{u_V}{V} = \frac{0.05\text{ gal}}{1.15\text{ gal}} = 4.35\%$, $\frac{u_t}{t} = \frac{0.1\text{ s}}{33.0\text{ s}} = 0.303\%$, and $\frac{u_{\dot{V}}}{\dot{V}} = \frac{u_R}{R} = \frac{0.09113\text{ gpm}}{2.091\text{ gpm}} = 4.35\%$.
  - Notice that the relative uncertainty of $\dot{V}$ is the same as that of V (to three significant digits). If an engineer needs to improve the accuracy of the volume flow rate, he or she should concentrate on improving the accuracy of the measurement of *volume*, not time, since the volume measurement dominates the error in $\dot{V}$.
    - As a case in point, suppose the experimenters used a more precise instrument to measure the elapsed time, so that t = 33.00 s ± 0.05 s (a factor of two improvement in the uncertainty of the time measurement). The final result does not change, i.e., $\boxed{\dot{V} = 2.09 \pm 0.091 \frac{\text{gal}}{\text{min}}}$ with 95% confidence.
    - However, suppose the precision of the *volume* measurement were improved by a factor of two such that V = 1.150 ± 0.025 gal. The final result would change to $\boxed{\dot{V} = 2.09 \pm 0.046 \frac{\text{gal}}{\text{min}}}$ with 95% confidence – a significant improvement.
  - In examples such as this, in which one of the measurements has a much larger relative uncertainty compared to the others, the RSS uncertainty is only slightly larger than the maximum relative uncertainty.
  - In cases in which most of the relative uncertainties are similar in magnitude, and/or if there are *many* measured variables contributing to the equation (here there are only two variables), the RSS uncertainty may be significantly different than any single component uncertainty.

Combining Elemental Uncertainties: RSS Uncertainty Analysis

- The root-of-the-sum-of-the-squares (RSS) concept is also useful when one needs to combine ***elemental uncertainties***, defined as ***precision uncertainties, bias uncertainties, calibration uncertainties, etc.***
- Consider several elemental uncertainties u_1, u_2, ..., u_K for some measured quantity x, where K is the number of elemental uncertainties.
- The overall uncertainty for variable x is given the symbol u_x.
- To obtain an overall estimate of the uncertainty in the measurement of x, the standard convention is to use the RSS equation to account for each of the component elemental uncertainties, i.e., $\boxed{u_x = \sqrt{\sum_{i=1}^{i=K} u_i^2}}$.
- A common example is a case in which there are both bias (systematic) and precision (random) uncertainties. In such cases, the overall uncertainty is $\boxed{u_x = \sqrt{\left(u_{\text{systematic}}\right)^2 + \left(u_{\text{random}}\right)^2}}$.
- The following rules must be kept in mind in order to use the above estimate:
  - All elemental uncertainties must have the *same units*. (The term “units” is used loosely – fractional or relative uncertainties or percentages are often used in place of actual units.)
  - All elemental uncertainties must have been estimated with the *same confidence level* (the engineering standard is 95% – other confidence levels can be used, but only if done *consistently*).
  - All elemental uncertainties must be entered in +/- format (to avoid factor of 2 errors).

Example:

Given: Dozens of measurements are taken of the rotation speed, *N*, of a shaft in rpm (rotations per minute), using a stroboscope.

- The average of all the measurements is calculated to be 1734.2 rpm.
- The standard deviation of these measurements is calculated to be 1.45 rpm.
- The strobe manufacturer claims an accuracy of ±1.00 rpm on the strobe readout. (It is assumed that this accuracy is to standard 95% confidence level.)

To do: Estimate the overall uncertainty of the measurement in rpm, and write the value of the shaft rpm in standard engineering notation.

Solution:

- First, the standard deviation is converted to a 95% confidence level uncertainty, expressed as a +/- uncertainty, which we call the ***measurement uncertainty***, $u_{\text{measurement}} = \pm 2\sigma = \pm 2(1.45\text{ rpm}) = \pm 2.90\text{ rpm}$.
- We define $u_{\text{manufacturer}}$ as the ***manufacturer's uncertainty***, which is given: $u_{\text{manufacturer}} = \pm 1.00$ rpm.
- The two errors are combined with the RSS equation: $u_N = \sqrt{\sum_{i=1}^{i=K} u_i^2} = \sqrt{(u_{\text{measurement}})^2 + (u_{\text{manufacturer}})^2} = \sqrt{(2.90\text{ rpm})^2 + (1.00\text{ rpm})^2} = 3.068\text{ rpm}$.
- Note that all uncertainties are in ± form, and we round to three significant digits. Thus we write the overall uncertainty as $\boxed{u_N = \pm 3.07\text{ rpm}}$.
- Finally, the shaft rotation speed should be reported (in standard engineering notation) as: $\boxed{N = 1734.2 \pm 3.07\text{ rpm}}$, where we have rounded the uncertainty interval to two decimal places since *N* is precise to only one decimal place [our convention is to report the interval to one extra decimal place compared to the mean value].

Discussion: Since the manufacturer's uncertainty is only about a third of the measurement uncertainty in this example, the overall uncertainty is dominated by measurement uncertainty. For this case, then, a more accurately calibrated strobe would not significantly improve the overall uncertainty of the measurement. For example, if the manufacturer's uncertainty were reduced by almost a factor of two to ±0.60 rpm, the final result would become $N = 1734.2 \pm 2.96\text{ rpm}$, not much of an improvement.