Stress, Strain, and Strain Gages

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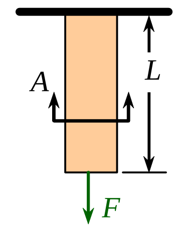
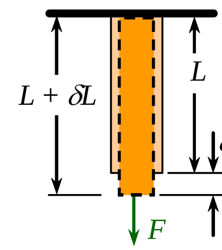
Introduction

- Stress and strain are important aspects of Mechanical Engineering, especially in structural design.
- In this learning module, we discuss stress and strain and their relationship, and how to measure them.

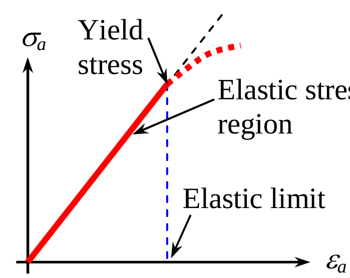
Definitions

- *Stress*
  - When a material is loaded with a force, the ***stress*** at some location in the material is defined as *the applied force per unit of cross-sectional area*.
  - For example, consider a wire or cylinder, anchored at the top, and hanging down. Some force *F* (for example, from a hanging weight) pulls at the bottom, as sketched, where *A* is the original cross-sectional area of the wire, and *L* is the original wire length.
  - In this situation, the material experiences a stress, called an ***axial stress***, denoted by the subscript *a*, and defined as $\boxed{\sigma_a = \frac{F}{A}}$.
  - Notice that *the dimensions of stress are the same as those of pressure* – force per unit area.





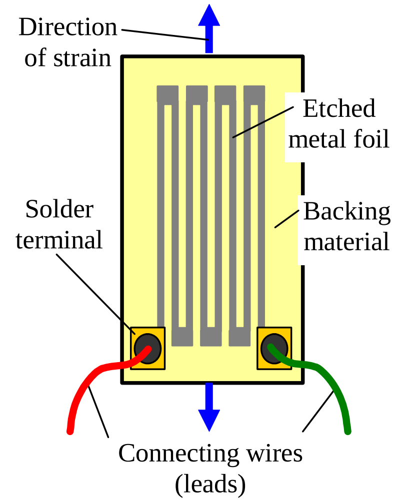
- *Strain*
  - In the above simple example, the wire stretches vertically as a result of the force. ***Strain*** is defined as *the ratio of increase in length to original length*.
  - Specifically, when force is applied to the wire, its length *L* increases by a small increment δL, while its cross-sectional area *A* decreases, as sketched.
  - In the axial direction (the direction of the applied force), ***axial strain*** ε_a is defined as $\boxed{\varepsilon_a = \frac{\delta L}{L}}$.
  - The dimensions of strain are unity – *strain is a nondimensional quantity*.
- *Hooke's law*
  - It turns out that ***for elastic materials, stress is linearly proportional to strain***.
  - Mathematically, this is expressed by ***Hooke's law***, which states $\boxed{\sigma_a = E\varepsilon_a}$, where *E* = ***Young's modulus***, also called the ***modulus of elasticity***.
  - Young's modulus is assumed to be constant for a given material.
  - Hooke's law breaks down when the strain gets too high. On a typical stress-strain diagram, Hooke's law applies only in the ***elastic stress region***, in which the loading is reversible. Beyond the ***elastic limit*** (or ***proportional limit***), the material starts to behave *irreversibly* in the ***plastic deformation region***, in which the stress vs. strain curve deviates from linear, and Hooke's law no longer holds, as sketched.
  - In this learning module, only the elastic stress region is considered.



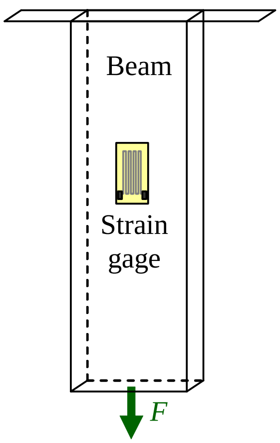
Wire resistance

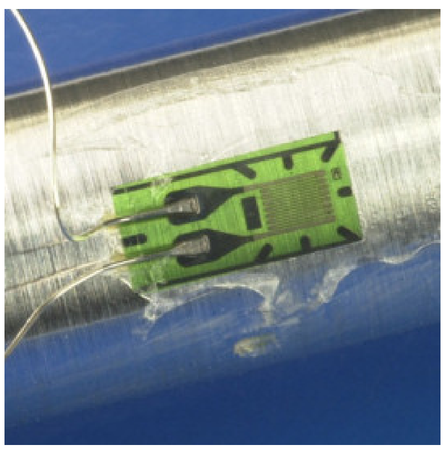
- The electrical resistance *R* of a wire of length *L* and cross-sectional area *A* is given by $\boxed{R = \frac{\rho L}{A}}$, where ρ is the ***resistivity*** of the wire material. (Do not confuse ρ with density, for which the same symbol is used.)
- The electrical resistance of the wire changes with strain:
  - As strain increases, the wire length *L* increases, which increases *R*.
  - As strain increases, the wire cross-sectional area *A* decreases, which increases *R*.
  - For most materials, as strain increases, the wire resistivity ρ also increases, which further increases *R*.
- The bottom line is that *wire resistance increases with strain*.
- In fact, it turns out that at constant temperature, wire resistance increases ***linearly*** with strain.
- Mathematically, $\boxed{\frac{\delta R}{R} = S\varepsilon_a}$, where *S* is the ***strain gage factor***, defined as $\boxed{S = \frac{\delta R / R}{\varepsilon_a}}$.
- *S* is typically around 2.0 for commercially available strain gages. *S is dimensionless*.

Strain gage



- The principle discussed above, namely that a wire's resistance increases with strain, is key to understanding how a strain gage works.
- The strain gage was invented by Ed Simmons at Caltech in 1936.
- A ***strain gage*** consists of a small diameter wire (actually an etched ***metal foil***) that is attached to a ***backing material*** (usually made of plastic) as sketched. The wire is looped back and forth several times to create an ***effectively longer wire***. The longer the wire, the larger the resistance, and the larger the change in resistance with strain.
- Here, four loops of metal foil are shown, providing an effective total foil length *L* that is eight times greater than if a single wire, rather than a looping pattern, were used. Commercially available strain gages have even more loops than this. The ones used in our lab have *six* loops.
- The direction of the applied strain is indicated on the sketch. The connecting wires or ***leads*** go to an electronic circuit (discussed below) that measures the change in resistance.
- Consider a beam undergoing axial strain; the strain is to be measured.
- A strain gage is glued to the surface of the beam, with the long sections of the etched metal foil aligned with the applied axial strain as sketched below left (the strain gage is mounted on the front face of the beam).
- As the surface stretches (strains), the strain gage stretches along with it. The resistance of the strain gage therefore increases with applied strain. Assuming the change in resistance can be measured, the strain gage provides a method for measuring strain.
- Other practical applications are shown below – a strain gage glued (rather sloppily) onto a cylindrical rod, and a strain gage mounted on a re-bar, which is then encased in concrete, used to measure shrinkage and to monitor the strain on structural components in bridges, buildings, etc.





Typical strain gage values

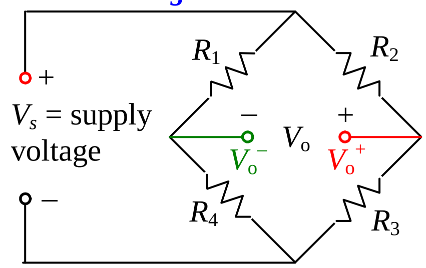
- Here are some typical values for resistance, strain gage factor, and strain, along with the predicted values of change in resistance:
  - The electrical resistance *R* of a commercial strain gage (with no applied strain) is typically either 120 Ω or 350 Ω.
  - The most widely used commercially available strain gages have R = 120 Ω.
  - The strain gage factor *S* of the metal foil used in strain gages is typically around 2.0.
  - In typical engineering applications with metal beams, the range of axial strain is $10^{-6} < \varepsilon_a < 10^{-3}$.
- Using these limits and the above equation for change in resistance as a function of strain and strain gage factor, $\delta R = RS\varepsilon_a$, and the typical range of δR is $(120\ \Omega)(2.0)(10^{-6}) < \delta R < (120\ \Omega)(2.0)(10^{-3})$, or 0.00024 Ω < δR < 0.24 Ω.
- Notice how small δR is!
- For a typical 120 Ω strain gage, the range of fractional change in resistance is $2 \times 10^{-6} < \delta R/R < 2 \times 10^{-3}$.
- This is the main problem when working with strain gages: We cannot use a simple ohm meter to measure the change in resistance, because $\delta R/R$ is so small. Most ohm meters do not have sufficient resolution to measure changes in resistance that are 3 to 6 orders of magnitude smaller than the resistance itself.

Strain gage electronics

Since $\delta R/R$ is very small and difficult to measure directly, electronic circuits must be designed to measure the *change* in resistance rather than the resistance itself. Fortunately, there are circuits available to do just that.

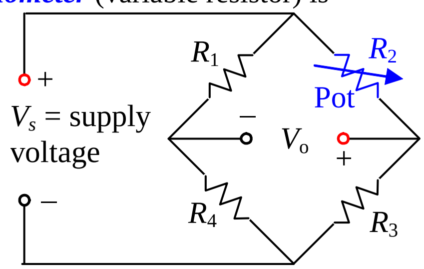
The Wheatstone bridge

- A clever circuit to measure very small changes in resistance is called a ***Wheatstone bridge***.
- A schematic diagram of a simple Wheatstone bridge circuit is shown to the right.
- As seen in the sketch, a DC supply voltage is supplied (top to bottom) across the bridge, which contains four resistors (two parallel legs of two resistors each in series).
- The output voltage is measured across the legs in the middle of the bridge.
- In the analysis here, it is assumed that the measuring device (voltmeter, oscilloscope, computerized digital data acquisition system, etc.) used to measure output voltage V_o has an *infinite input impedance*, and therefore has *no effect on the circuit*.



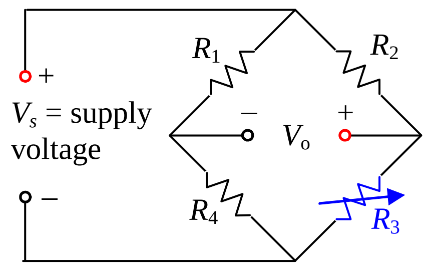
- Output voltage $V_o = V_o^+ - V_o^-$ is calculated by analyzing the circuit. Namely, $V_o = V_s \dfrac{R_3 R_1 - R_4 R_2}{(R_2 + R_3)(R_1 + R_4)}$.

  [This equation is "*exact*" – no approximations of small change in resistance were made in its derivation.]
- How does the Wheatstone bridge work? Well, if all four resistors are identical ($R_1 = R_2 = R_3 = R_4$), the bridge is ***balanced*** since the same current flows through the left leg and the right leg of the bridge. *For a balanced bridge, $V_o = 0$.*
- More generally (as can be seen from the above equation), a Wheatstone bridge can be balanced even if the resistors do *not* all have the same value, so long as the numerator in the above equation is zero, i.e., if $R_3 R_1 = R_4 R_2$. Or, expressed as ratios, the bridge is balanced if $\dfrac{R_1}{R_2} = \dfrac{R_4}{R_3}$.
- In practice, the bridge will *not* be balanced automatically, since "identical" resistors are not actually identical, with resistance varying by up to several percent. Thus, a ***potentiometer*** (variable resistor) is sometimes applied in place of one of the resistors in the bridge so that minor adjustments can be made in order to balance the bridge.
  - An arrow through the resistor indicates that its resistance can *vary*, as sketched to the right.
  - In this circuit, resistor R_2 was arbitrarily chosen to be replaced by a potentiometer, but any of the four resistors could have been used instead.



Quarter bridge circuit

- To measure strain, one of the resistors, in this case R_3, is replaced by the strain gage, as sketched to the right. (Note that one of the other resistors may still be a potentiometer rather than a fixed resistor, but that will not be indicated on the circuit diagrams to follow.)
- Again, an arrow through the resistor indicates that its resistance can vary – this time because R_3 is an active strain gage, not a potentiometer.
- With only *one* out of the four available resistors substituted by a strain gage, as in the above schematic, the circuit is called a ***quarter bridge circuit***.



R_3 = strain gage

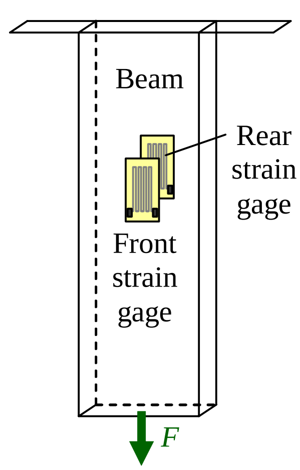
- The output voltage V_o is calculated from Ohm's law, as previously, $V_o = V_s \dfrac{R_3 R_1 - R_4 R_2}{(R_2 + R_3)(R_1 + R_4)}$.
- Let $R_1 = R_2 = R_4 = 120\ \Omega$, and let the initial resistance of the strain gage (with no load) be $R_{3,\text{initial}} = 120\ \Omega$.
- The bridge is therefore initially *balanced* when $R_3 = R_{3,\text{initial}}$, since $R_{3,\text{initial}} R_1 - R_4 R_2 = 0$, and V_o is thus zero.

Unbalanced quarter bridge circuit - to measure strain

- In normal operation, the Wheatstone bridge is initially balanced as above. Now suppose strain is applied to the strain gage, such that its resistance changes by some small amount δR_3. In other words, R_3 changes from $R_{3,\text{initial}}$ to $R_{3,\text{initial}} + \delta R_3$.
- Under these conditions the bridge is ***unbalanced***, and the resulting output voltage V_o is *not* zero, but can be calculated as $V_o = V_s \dfrac{\left(R_{3,\text{initial}} + \delta R_3\right)R_1 - R_4 R_2}{\left(R_2 + R_{3,\text{initial}} + \delta R_3\right)\left(R_1 + R_4\right)}$.
- We simplify the numerator by applying the initial balance equation, $R_{3,\text{initial}}R_1 - R_4R_2 = 0$, yielding $V_o = V_s \dfrac{\delta R_3 \cdot R_1}{\left(R_2 + R_{3,\text{initial}} + \delta R_3\right)\left(R_1 + R_4\right)}$. [This equation is *exact* only if the bridge is initially balanced.]
- We simplify the denominator by recognizing, as pointed out previously, that the change in resistance of a strain gage is very small; in other words, $\delta R_3/R_{3,\text{initial}} << 1$. This yields $V_o \approx V_s \dfrac{\delta R_3 \cdot R_1}{\left(R_2 + R_{3,\text{initial}}\right)\left(R_1 + R_4\right)}$.
- We apply the relationship derived earlier for change in resistance of a strain gage as a function of axial strain, resistance, and strain gage factor, namely, $\delta R_3 = R_{3,\text{initial}} S \varepsilon_a$. After some algebra, $\varepsilon_a \approx \dfrac{V_o}{V_s}\dfrac{1}{S}\dfrac{\left(R_2 + R_{3,\text{initial}}\right)^2}{R_2 R_{3,\text{initial}}}$.
- Furthermore, since $R_2 = R_{3,\text{initial}}$ (e.g., both are 120 Ω), this reduces to $\varepsilon_a \approx 4\dfrac{V_o}{V_s}\dfrac{1}{S}$ or $V_o \approx \dfrac{\varepsilon_a S}{4} V_s$.
- The significance of this result is this:
  For constant supply voltage V_s and constant strain gage factor S, axial strain at the location of the strain gage is a *linear function* of the output voltage from the Wheatstone bridge circuit.
- Even more significantly:
  For known values of S and V_s, the actual value of the strain can be calculated from the above equation after measurement of output voltage V_o.

- **Example**:
  Given: A standard strain gage is used in a quarter bridge circuit to measure the strain of a beam in tension. The strain gage factor is $S = 2.0$, and the supply voltage to the Wheatstone bridge is $V_s = 5.00$ V. The bridge is balanced when no load is applied. Assume all resistors are equal when the strain gage circuit is initially balanced with no load. For a certain non-zero load, the measured output voltage is $V_o = 1.13$ mV.
  To do: Calculate the axial strain on the beam.
  Solution:
  - We apply the above equation for axial strain for a quarter bridge circuit, yielding
    $$\varepsilon_a \approx 4\frac{V_o}{V_s}\frac{1}{S} = 4\frac{1.13\text{ mV}}{5.00\text{ V}}\frac{1}{2.0}\left(\frac{1\text{ V}}{1000\text{ mV}}\right) = 0.000452.$$
  - Since strain is such a small number, it is common to report strain in units of ***microstrain*** (μstrain), defined as the strain times 10^6. Note that strain is dimensionless, so microstrain is a dimensionless unit.
  - The unit conversion between strain and microstrain, expressed as a dimensionless ratio, is (10^6 microstrain/strain). Thus,
    $$\varepsilon_a = 0.000452\left(\frac{10^6\ \mu\text{strain}}{\text{strain}}\right) = 452\ \mu\text{strain}.$$
  - Finally, keeping to two significant digits (since S is given to only two digits), $\varepsilon_a = 450\ \mu\text{strain}$.

  Discussion: It is also correct to give the final answer as $\varepsilon_a = 0.00045$.

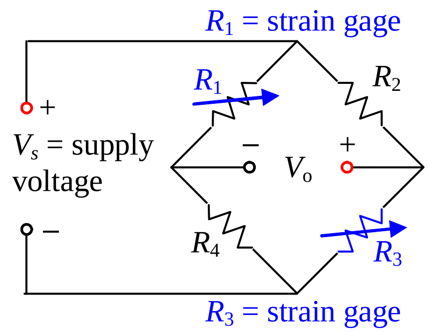


Half bridge circuit

- Suppose we mount *two* active strain gages on the beam, one at the front and one at the back as sketched to the right.
- Also suppose that *both* strain gages are put into the Wheatstone bridge circuit, as shown in the circuit diagram below, noting that resistors R_1 and R_3 have been

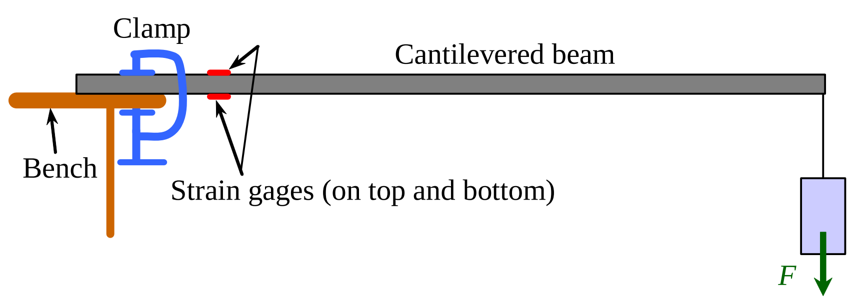
replaced by the two strain gages.

- Since *half* of the four available resistors in the bridge are strain gages, this is called a ***half bridge circuit***.
- After some algebra, assuming that both strain gage resistances change identically as the strain is applied, it can be shown that $\varepsilon_a \approx \frac{V_o}{V_s}\frac{1}{2S}\frac{\left(R_2 + R_{3,\text{initial}}\right)^2}{R_2 R_{3,\text{initial}}}$.
- Furthermore, since $R_2 = R_{3,\text{initial}} = 120\ \Omega$, the above equation reduces to $\boxed{\varepsilon_a \approx 2\frac{V_o}{V_s}\frac{1}{S}}$ or $\boxed{V_o \approx \frac{\varepsilon_a S}{2}V_s}$.

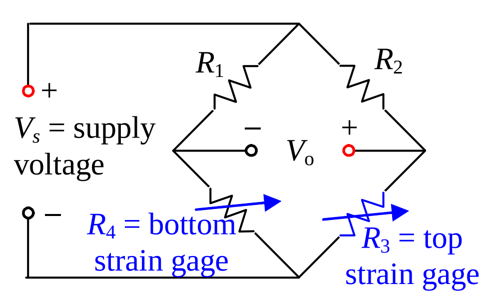


- Compared to the quarter bridge circuit, *the half bridge circuit yields twice the output voltage for a given strain*. We say that *the* ***sensitivity*** *of the circuit has improved by a factor of two*.
- You might ask why R_1 (rather than R_2 or R_4) was chosen as the resistor to replace with the second strain gage. It turns out that R_1 is used for the second strain gage if its strain is of the same sign as that of R_3.
- To prove the above statement, suppose all four resistors are strain gages with initial values $R_{1,\text{initial}}$, $R_{2,\text{initial}}$, etc. The corresponding changes in resistance due to applied strain are δR_1, δR_2, etc. It can be shown (via application of Ohm's law, and neglecting higher-order terms as previously) that the output voltage varies as $\boxed{\frac{V_o}{V_s} \approx \frac{R_{2,\text{initial}}R_{3,\text{initial}}}{\left(R_{2,\text{initial}} + R_{3,\text{initial}}\right)^2}\left(\frac{\delta R_1}{R_{1,\text{initial}}} - \frac{\delta R_2}{R_{2,\text{initial}}} + \frac{\delta R_3}{R_{3,\text{initial}}} - \frac{\delta R_4}{R_{4,\text{initial}}}\right)}$. [This equation is approximate – assumes initially balanced bridge and small changes in resistance.]
- As can be seen, the terms with δR_1 and δR_3 are of *positive* sign, and therefore contribute to a *positive* output voltage when the applied strain is positive (strain gage in tension).
- However, the terms with δR_2 and δR_4 are of *negative* sign, and therefore contribute to a *negative* output voltage when the applied strain is positive (strain gage in tension).
- In the above beam example, in which both strain gages measure the *same* strain, it is appropriate to choose R_1 for the second strain gage. If R_2 or R_4 had been chosen instead, the output voltage would not change at all as strain is increased, because of the signs in the above equation. (The change in resistance of the two strain gages would cancel each other out!)

Example – a cantilever beam experiment

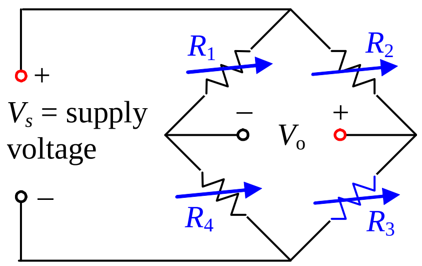


- As an example, consider a simple lab experiment. A cantilevered beam is clamped to the lab bench, and a weight is applied at the end of the beam as sketched to the right. A strain gage is attached on the top surface of the beam, and another is attached at the bottom surface, as shown.
- As the beam is strained due to the applied force, ***the top strain gage is stretched*** (positive axial strain), but ***the bottom strain gage is compressed*** (negative axial strain).
- If the beam is symmetric in cross section, and if the two strain gages are identical, the two strain gages have approximately the same magnitude of change in resistance, but opposite signs, i.e., $\delta R_{\text{bottom}} = -\delta R_{\text{top}}$.
- In this case, if R_1 and R_3 were chosen for the two strain gages in the bridge circuit, the Wheatstone bridge would remain balanced for *any* applied load, since δR_1 and δR_3 would cancel each other out.
- In this example, the half bridge circuit should be constructed with pairs of resistors that have *opposite signs in the above general equation* – the choices are R_1 and R_2, R_1 and R_4, R_2 and R_3, or R_3 and R_4 as the two resistors to be substituted by the strain gages.
- An example circuit for this simple experiment uses R_3 for the top strain gage and R_4 for the bottom strain gage, with the Wheatstone bridge circuit wired as sketched to the right.



- Circuit analysis for this case yields $\varepsilon_a \approx 2\frac{V_o}{V_s}\frac{1}{S}$ or $V_o \approx \frac{\varepsilon_a S}{2} V_s$.
- Compared to the quarter bridge circuit, the voltage output of this half bridge circuit (with two active strain gages) is *twice* that of the quarter bridge circuit (with only one active strain gage), all else being equal.
- For any system, ***sensitivity*** is defined as (change in output) / (change in input). In this case, the output is the voltage V_o, and the input is the axial strain being measured.
- Thus, we conclude: The sensitivity of a half bridge Wheatstone bridge circuit is *twice* that of a quarter bridge Wheatstone bridge circuit.

Full bridge circuit



- If we substitute strain gages for ***all four*** resistors in a Wheatstone bridge, the result is called a ***full bridge circuit***, as sketched to the right.
- Warning: *You need to be very careful with the signs when wiring a full bridge circuit*!
- If the wiring is done properly (e.g., R_1 and R_3 have positive strain, while R_2 and R_4 have negative strain), the sensitivity of the full bridge circuit is four times that of a quarter bridge circuit, $\varepsilon_a \approx \frac{V_o}{V_s}\frac{1}{S}$ or $V_o \approx \varepsilon_a S V_s$.
- In general, we define ***n*** as the number of active gages in the Wheatstone bridge:
  - o $n = 1$ for a quarter bridge
  - o $n = 2$ for a half bridge
  - o $n = 4$ for a full bridge
- Then the strain can be generalized to $\varepsilon_a \approx \frac{4}{n}\frac{V_o}{V_s}\frac{1}{S}$ or $V_o \approx \frac{n}{4}\varepsilon_a S V_s$.
- One cautionary note: In derivation of the above equation, it is assumed that positive strain gages (R_1 and R_3) are chosen for positive strain (tension), and negative strain gages (R_2 and R_4) are chosen for negative strain (compression). If instead we were to wire the circuit such that the positive gages are in compression and the negative gages are in tension, a negative sign would appear in the above equation.
- On a final note, it is not always necessary to initially balance the bridge. In other words, suppose there is some initial non-zero value of bridge output voltage, namely $V_{o,\text{reference}} \neq 0$. This voltage represents the ***reference output voltage*** at some initial conditions of the experiment, which may not necessarily even be zero strain.
- We can still calculate the strain by using the output voltage *difference* rather than the output voltage itself, $\varepsilon_a \approx \frac{4}{n}\frac{\left(V_o - V_{o,\text{reference}}\right)}{V_s}\frac{1}{S}$ or $V_o \approx V_{o,\text{reference}} + \frac{n}{4}\varepsilon_a S V_s$.
- In the lab, we use voltmeters with a "**REL**" button, which stands for "relative" voltage. [On some voltmeters, the **REL** button is indicated by triangular symbol "**Δ**" instead.]
- Under conditions of zero strain with a slightly unbalanced bridge, the reference output voltage is not zero ($V_{o,\text{reference}} \neq 0$). However, pushing the **REL** button causes the voltmeter to read all subsequent voltages *relative* to $V_{o,\text{reference}}$. In other words, the voltmeter reads $V_o - V_{o,\text{reference}}$ instead of V_o itself. We effectively "trick" the voltmeter into showing zero voltage for zero strain.
- ***Warning***: The **REL** button can be dangerous if you forget to turn it off, because *all subsequent voltages are displayed relative to the voltage input at the time the* ***REL*** *button was pushed*.