Today, we will:

- Continue to discuss isentropic, adiabatic duct flow & derive equations for ducts of variable area
- Describe variable area duct behavior for subsonic, sonic, and supersonic flow
- Discuss the significance of a ***throat*** (minimum area) in a variable area duct
- Begin to discuss so-called ***critical properties*** of compressible flow

Steady, isentropic, adiabatic, one-dimensional duct flow of an ideal gas (continued)

Last lecture, we wrote the following equations:

From the adiabatic condition (first law),

$$\boxed{dh = -VdV} \quad (1)$$

From one of the *Tds* equations,

$$\boxed{dh = \frac{dP}{\rho}} \quad (2)$$

Equating (1) and (2),

$$\boxed{-VdV = \frac{dP}{\rho}} \quad (3)$$

From conservation of mass (by differentiating),

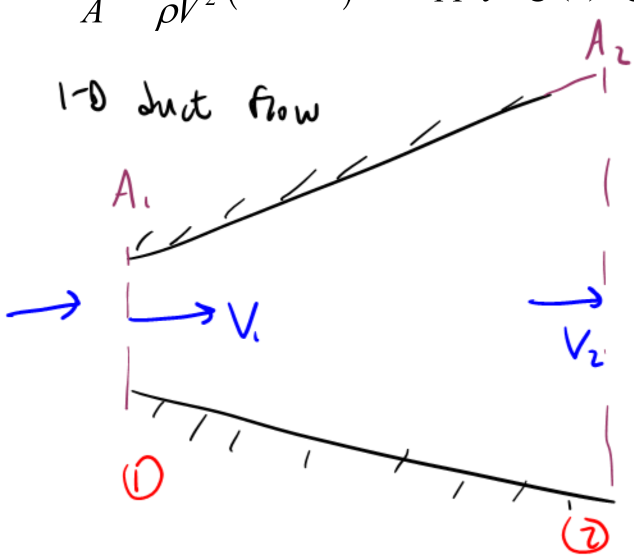
$$\boxed{\frac{dA}{A} + \frac{dV}{V} + \frac{d\rho}{\rho} = 0} \quad (4)$$

Plugging Eq. (3) into (4) and rearranging,

$$\frac{dA}{A} - \frac{dP}{\rho V^2} + \frac{d\rho}{\rho} = 0 \text{ or, rearranging, } \boxed{\frac{dA}{A} = \frac{dP}{\rho}\left(\frac{1}{V^2} - \frac{d\rho}{dP}\right)} \quad (5)$$

Finally, a little more manipulation using the speed of sound and Mach number,

$$\frac{dA}{A} = \frac{dP}{\rho V^2}\left(1 - M^2\right) \text{ or, applying (3) again, } \boxed{\frac{dV}{V} = \frac{1}{\left(M^2 - 1\right)}\frac{dA}{A}} \quad (6)$$



adiabatic

Subsonic flow $Ma < 1$,

$\therefore M^2 - 1$ is $\ominus$

$$\frac{dV}{V} = \ominus \text{number} \frac{dA}{A}$$

if $A \uparrow$, $V \downarrow$

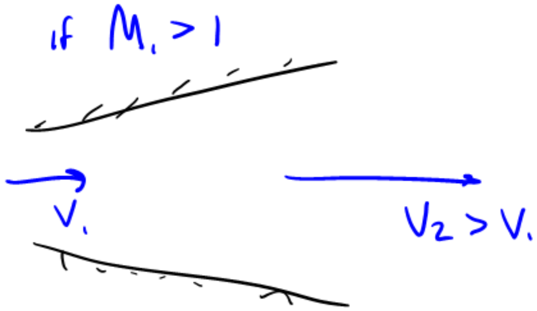
if $A \downarrow$, $V \uparrow$

If <u>supersonic flow</u> $M > 1$ $M^2 - 1$ is $\oplus$

$$\therefore \frac{dV}{V} = \oplus \text{ number } \frac{dA}{A}$$

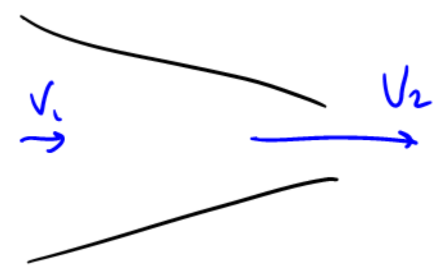
$\boxed{\text{if } A\uparrow,\ V\uparrow \quad \text{if } A\downarrow,\ V\downarrow}$ ✩

if $M_1 > 1$



CONVERGING DUCTS

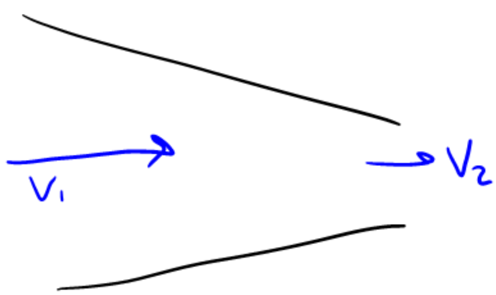
Subsonic



$\boxed{M<1 \quad V\uparrow \text{ as } A\downarrow}$

✩ $\boxed{M \text{ also goes } \uparrow \text{ as } V\uparrow}$

Supersonic



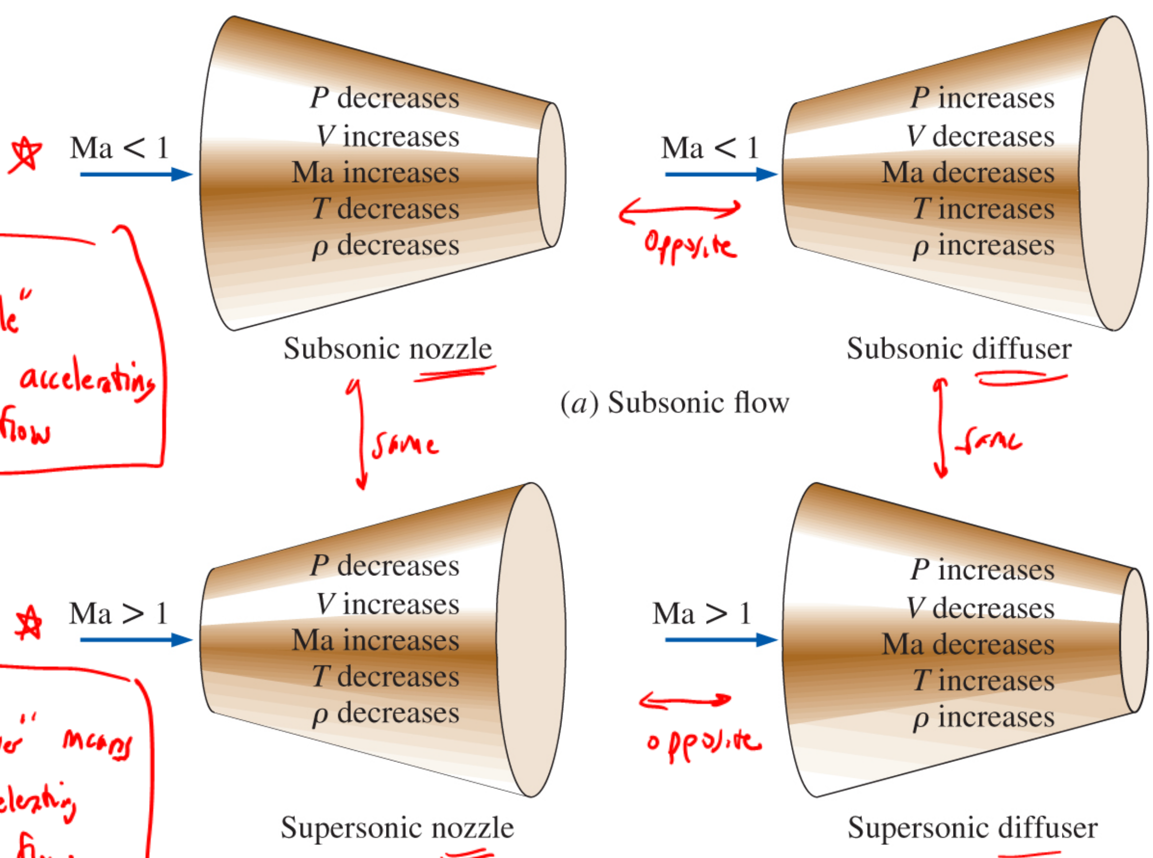
$\boxed{M>1 \quad V\downarrow \text{ as } A\downarrow}$

✩ $\boxed{M \text{ also goes } \downarrow \text{ as } V\downarrow}$

T, P, ρ, etc. also change oppositely for subsonic vs. supersonic flows

<u>See summary Figure from CC book</u>

Summary of the opposite behavior of subsonic and supersonic variable area duct flows:



(*a*) Subsonic flow

(*b*) Supersonic flow

"nozzle" means accelerating the flow

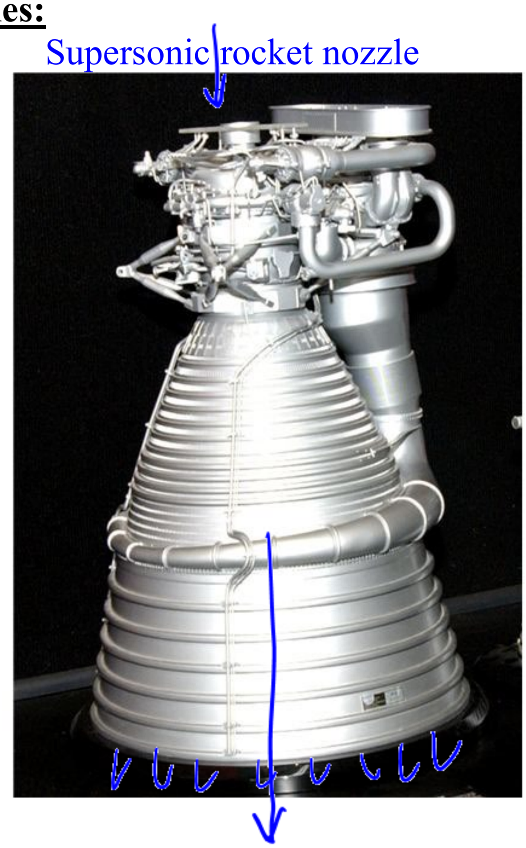
"Diffuser" means decelerating the flow

Figure 12-11 from Cengel & Cimbala

Comparison of subsonic and supersonic nozzles:

Subsonic water nozzle

Supersonic rocket nozzle



What happens when $M=1$? (sonic)

$$\frac{dV}{V} = \frac{1}{M^2-1}\,\frac{dA}{A}$$

$M^2-1 = 0$ → $\infty \times 0$

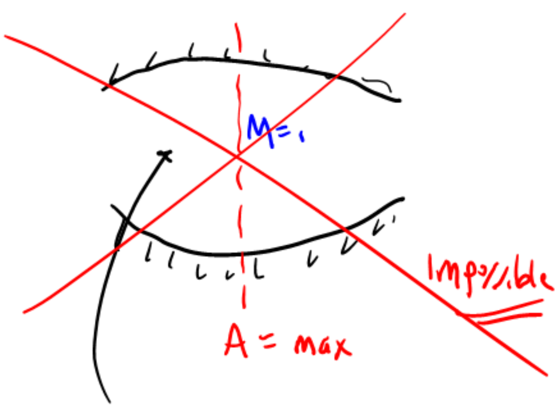
$\frac{dA}{A}$ must be zero @ $M=1$

$$\frac{dA}{A} = \underbrace{(M^2-1)}_{0}\,\frac{dV}{dV}$$

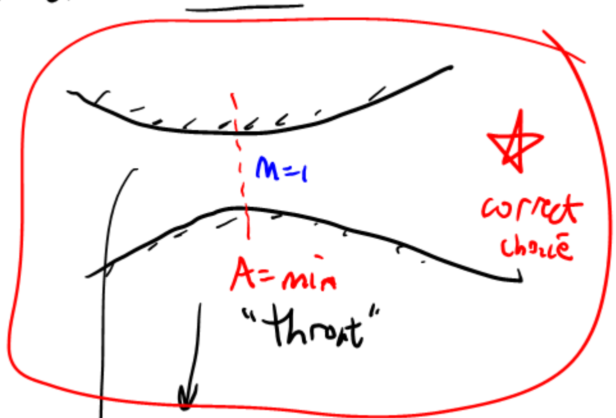
$\boxed{\frac{dA}{A}=0}$ @ $M=1$

$(dA=0)$

∴ A must be either a maximum or a minimum @ $M=1$



or



if $M>1$, $M\uparrow$
so M cannot $\rightarrow 1$

if $M<1$ $M\downarrow$
so M cannot $\rightarrow 1$

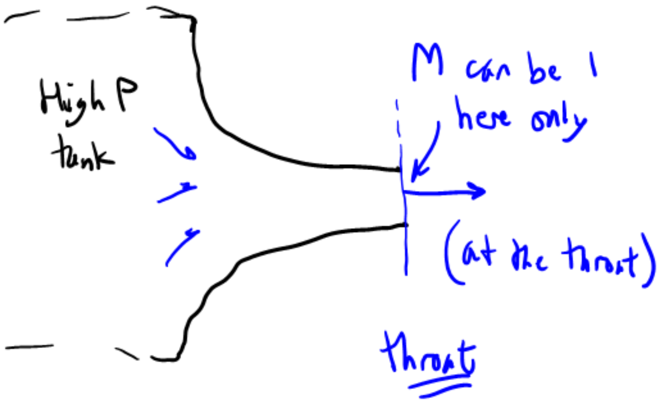
if $M<1$ $M\uparrow$, ∴ M can $\rightarrow 1$
@ throat

if $M>1$ $M\downarrow$ ∴ M can $\rightarrow 1$
@ throat

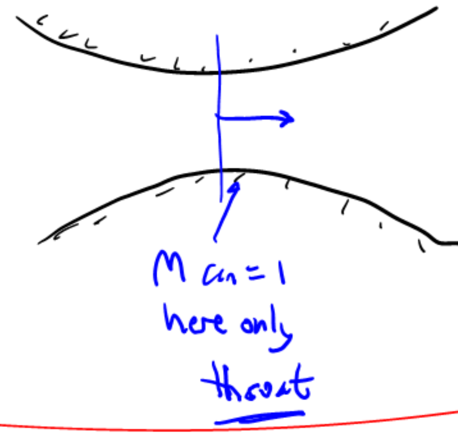
CONCLUSION: $\boxed{\text{Area } A \text{ must be a minimum in order for } M=1 \text{ for 1-D isentropic adiabatic duct flow of variable area}}$

There must be a throat

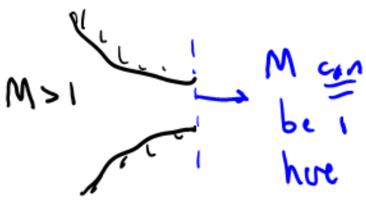
Converging duct



converging-diverging duct



For adiabatic isentropic flow, the duct must be converging to get to M=1 for both subsonic & supersonic upstream flow

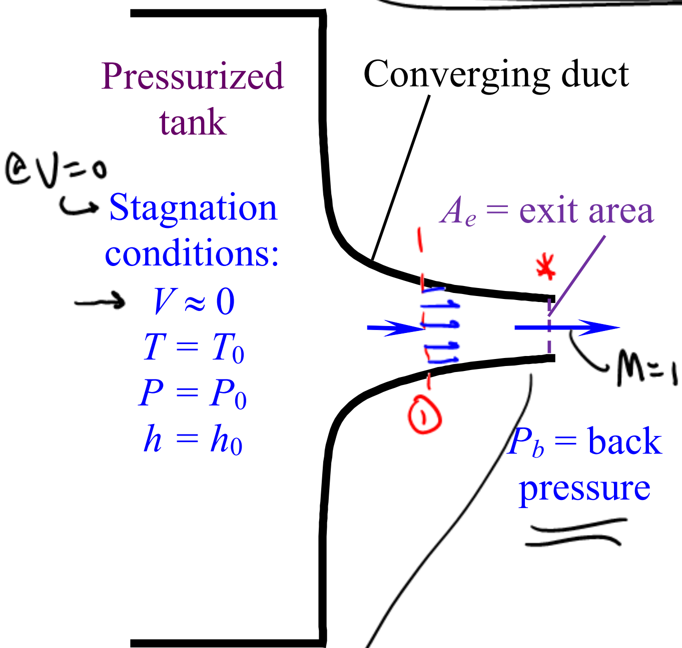


Comment: The presence of a throat does not guarantee that M=1 @ the throat!

(depends on upstream & downstream pressures)

But If M=1 somewhere in the duct, then M=1 @ the throat

✰ CRITICAL PROPERTIES



Consider case where P_b is low enough that $M=1$ @ the throat

$\boxed{\text{When } M=1 \text{ @ throat, flow is choked}}$ ✰

Superscript * means sonic conditions → $M=1$

we also call these critical conditions ✰

@ *, let $T = T^*$
$P = P^*$
$a = a^*$
$A = A^*$

T^* = "critical temperature"

We can define & use these * properties in our calculations even if the flow is not sonic @ the throat ✰

$$a^* = \text{critical speed of sound} = \sqrt{\gamma R T^*}$$

Exception $M^* \neq 1$

instead M^* = characteristic Mach # (in Anderson's notation)

$$\boxed{M^* = \frac{V}{a^*}} ✰ \qquad \left[M = \frac{V}{a}\right]$$

It can be shown that for an ideal gas

$$\boxed{M^2 = \frac{2}{\frac{\gamma+1}{M^{*2}} - (\gamma-1)}} ✰$$