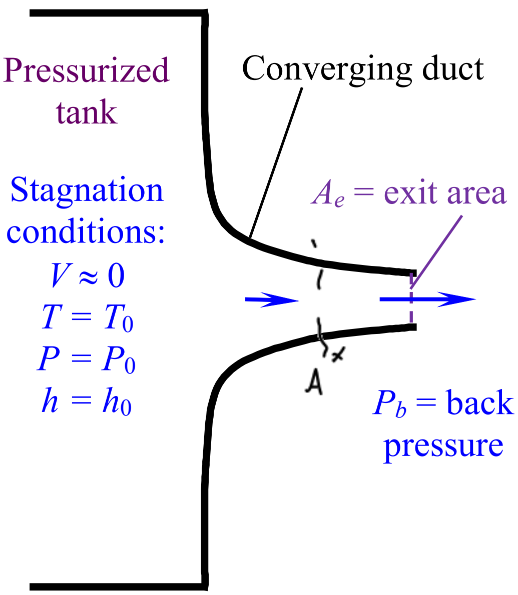
**Today, we will**:

- Generate the ***area ratio vs. Mach number relationship*** for steady, adiabatic, isentropic one-dimensional duct flow of an ideal gas
- Show various ways to solve the resulting ***implicit equation***
- Start an example problem – converging duct, calculate *M* and *P* at various *x* locations

**The area ratio vs. Mach number relationship** (for steady, adiabatic, isentropic one-dimensional duct flow of an ideal gas):

Consider again our converging nozzle flow:




From the previous lecture (also see equation sheet), we had:

General case: $$\dot{m} = P_0 A M \sqrt{\frac{\gamma}{RT_0}}\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{-(\gamma+1)}{2(\gamma-1)}} \quad (4)$$

Choked case: $$\dot{m} = \dot{m}_{max} = P_0 A^* \sqrt{\frac{\gamma}{RT_0}}\left(\frac{\gamma+1}{2}\right)^{\frac{-(\gamma+1)}{2(\gamma-1)}} \quad (5)$$

Q How to calc M as a fnc. of A through the nozzle?

A – combine (4) & (5) & manipulate

At any location, x → some area A

Eq. (4) = (5) $$\cancel{P_0} A M \cancel{\sqrt{\frac{\gamma}{RT_0}}}\left(1+\frac{\gamma-1}{2}M^2\right)^{\{\}} = \cancel{P_0} A^* \cancel{\sqrt{\frac{\gamma}{RT_0}}}\left(\frac{\gamma+1}{2}\right)^{\{\}}$$

Solve for $$\boxed{\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

steady
adiabatic
isentropic
1-D approx
ideal gas

This eq. applies even if $A^*$ does not exist in the flow!

Check: verify @ $M=1$ $\frac{A}{A^*} = \frac{1}{1}\left[\frac{2}{\gamma+1}\underbrace{\left(1+\frac{\gamma-1}{2}\right)}_{1}\right]^{\circ}$

$\boxed{\frac{A}{A^*} = 1}$ ✓

If we know $\frac{A}{A^*}$ how to calc $M$? IMPLICIT ☆

If we know $M \rightarrow$ easy to calc $\frac{A}{A^*}$ — explicit

## Side Notes: How to solve an implicit equation

**Example: how to solve for Mach number *M* at a given value of *A*/*A**?**

**Given**: The area ratio vs. Mach number relation for steady, adiabatic, isentropic, one-D duct flow of an ideal gas:

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad ✩ \qquad (1)$$

✩

This equation is ***explicit*** if *M* is known and we are solving for area ratio. But the equation is ***implicit*** if we want to solve for *M* at a given value of area ratio.

**To do**: Discuss various ways to solve for Mach number *M* at a given value of $A/A^*$.

**Solution**: **To be completed in class.**

Example, suppose $\boxed{\frac{A}{A^*} = 1.50}$ → CALC. M for air ($\gamma = 1.40$)

Note: $\frac{A}{A^*} > 1$

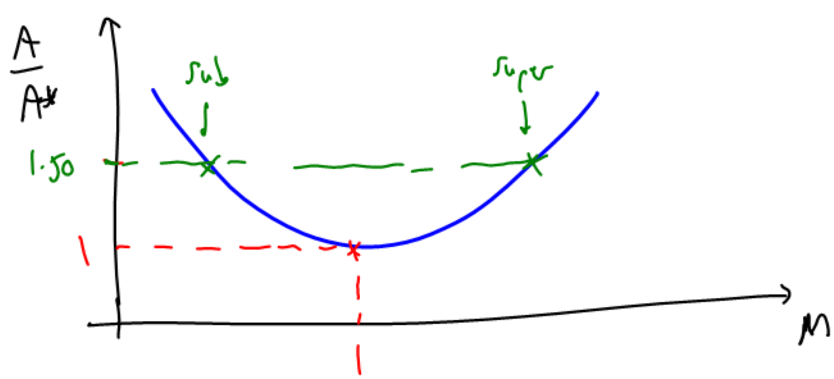
2 roots
- one subsonic ($M<1$)
- one supersonic ($M>1$)

if $A = A^*$ → $M = 1$

WAYS TO SOLVE THIS IMPLICIT EQ

(1) Fancy calculator → solve for you

(2) Graphically → Plot (1) & pick off M from graph



| | A | B | C | D | E | F | G | H | I | J | K | L | M | N |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Isentropic Mach numbers in a nozzle using the area ratio-Mach number equation, calculated various ways, including the False Position Method, What-IF Analysis, Ne** | | | | | | | | | | | | | |
| 2 | **J. M. Cimbala** | | | | | | | | | | | | | |
| 4 | **Constants:** | | | | | | | | | | | | | |
| 5 | | $\gamma$ = | 1.4 | - | ratio of specific heats | | | | | | | | | |
| 6 | | $A^*$ = | 0.006223 | m$^2$ | throat area | | | | | | | | | |
| 7 | | $A$ = | 0.0093345 | m$^2$ | duct area | | | | | | | | | |
| 8 | | exponent = | 3 | | exponent for $A/A^*$ vs Mach number equation | | | | | | | | | |
| 9 | | | | | | | | | | | | | | |
| 10 | **Goal seek method:** | | | | | | | | | | | | | |
| 11 | | **Goal =** | 1.5 | Goal for area ratio - need to find the $M$ that gives this value. | | | | | | | | | | |
| 12 | | 0.430367 | 1.4997111 | Data - What if Analysis - Goal Seek | | | | | | | | | | |
| 13 | | | | | | | | | | | | | | |
| 14 | **False Position Method ("smart trial and error using interpolation"):** | | | | | | | | | | | | | |
| 15 | | **Subsonic** | | | | **Supersonic** | | | | | | | | |
| 16 | | ***M*** | ***A/A**** | | | ***M*** | ***A/A**** | | | | | | | |
| 17 | **Guess:** | 0.3 | 2.0350653 | first guess | | 2 | 1.6875 | first guess | | | | | | |
| 18 | **Guess again:** | 0.7 | 1.0943727 | second guess | | 3 | 4.234568 | second guess | | | | | | |
| 19 | **Interpolate:** | 0.52752 | 1.2905775 | From here on, interpolate | | 1.9263859 | 1.588552 | From here on, interpolate with the previous two values - converges fairly rapidly if | | | | | | |
| 20 | | 0.34342 | 1.807197 | | | 1.8904562 | 1.543489 | | | | | | | |
| 21 | | 0.452891 | 1.4415918 | | | 1.8557821 | 1.50194 | | | | | | | |
| 22 | | 0.435402 | 1.4861101 | | | 1.8541627 | 1.500046 | | | | | | | |
| 23 | | 0.429946 | 1.5008666 | | | 1.8541236 | 1.5 | | | | | | | |
| 24 | | 0.430266 | 1.499988 | | | 1.8541235 | 1.5 | | | | | | | |
| 25 | | 0.430262 | 1.5 | | | 1.8541235 | 1.5 | | | | | | | |
| 26 | | 0.430262 | 1.5 | | | 1.8541235 | 1.5 | | | | | | | |
| 27 | | 0.430262 | 1.5 | | | 1.8541235 | 1.5 | | | | | | | |
| 28 | | 0.430262 | 1.5 | | | 1.8541235 | 1.5 | | | | | | | |
| 29 | | 0.430262 | 1.5 | | | 1.8541235 | 1.5 | | | | | | | |
| 30 | | 0.430262 | 1.5 | | | 1.8541235 | 1.5 | | | | | | | |
| 31 | | | | | | | | | | | | | | |
| 32 | **Simple calculations to verify:** | | | | | | | | | | | | | |
| 33 | | ***M*** | ***A/A**** | | | ***M*** | ***A/A**** | | | | | | | |
| 34 | **Subsonic** | **0.430262** | **1.5** | | **Supersonic** | **1.8541235** | **1.5** | | | | | | | |
| 35 | | | | | | | | | | | | | | |
| 36 | **Newton's Method:** | | | | **Iteration scheme using Newton's method:** | | | | | | | | | |
| 37 | | ***A/A**** | **Guess *M*** | **Final *M*** | **Final *G*=0?** | ***G*** | ***G'*** | **New *M*** | ***G*** | ***G'*** | **New *M*** | ***G*** | ***G'*** | **New *M*** |
| 38 | **Subsonic** | 1.5 | 0.3 | 0.4302617 | 0 | -0.5350653 | 6.063881 | 0.388238 | -0.129499 | 3.46022 | 0.425663 | -0.012755 | 2.80819 | 0.430205 |
| 39 | **Supersonic** | 1.5 | 1.5 | 1.8541235 | 0 | 0.3238329 | -0.675958 | 1.979073 | -0.158445 | -1.370566 | 1.863467 | -0.010987 | -1.183136 | 1.854181 |

3) Excel → WHAT IF ANALYSIS

4) False Position Method = "Smart Trial & Error"

$\frac{A}{A^*} = 1.50$ is my goal

| | $M$ | $A/A^*$ from Eq. (1) |
|---|---|---|
| Guess 1 | 0.3 | 2.0351 |
| | $\square$? | → $\square$ 1.50 |
| Guess 2 | 0.7 | 1.0944 |
| "Smart" guess 3 | 0.52752 | → 1.2906 |

guess 3 → interpolate ✓

$$\square = 0.7 + \frac{1.50 - 1.0944}{2.0351 - 1.0944}(0.3 - 0.7)$$

$$\square = 0.52752$$

Interpolate with the previous 2 values to get next guess converges to $\boxed{M = 0.430262}$ ✓

5) Newton's method → utilize derivative (slope)

For function $G(M)$ → pick $G$ such that $G(M) = 0$ (goal) when $M$ is correct value

Procedure:

- Guess $M$
- Calc $G$ @ this $M$
- Calc $G'(M)$ (deriv.)
- New guess for $M$ $\boxed{M_{new} = M_{old} - \frac{G}{G'}}$ ✮
- Repeat until $G(M)=0$ → this is the correct $M$

✮

Here, let $$\boxed{G(M) = \frac{A}{A^*} - \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

$$G'(M) = 0 - \left(\frac{-1}{M^2}\right)\left[\quad\right]^{\frac{\gamma-1}{2(\gamma-1)}} + -\frac{1}{M} \text{ deriv. of this}$$

Product rule

Guess $M = 0.2$ → iterate $\underline{M = 0.43026}$

$M = 1.5$ → " $\underline{M = 1.85412}$

6) Computer programs → MATLAB, EES

7) COMPRESSIBLE FLOW CALCULATOR ✱ → USEFUL FOR VALIDATING YOUR CALCULATIONS ✱, But I would not rely on it

See LINK ON WEBSITE