Today, we will:

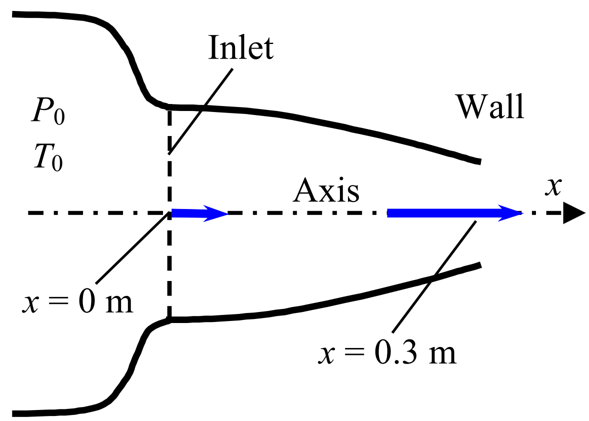
- Do an example problem – converging duct, calculate M and P at various x locations for the choked and non-choked case (Parts a and b).
- Examine what happens to the flow *downstream* of the exit of a converging nozzle

Example: Converging nozzle

Given: Air flows from a very large tank through a converging nozzle. The nozzle begins at $x = 0$. The outlet of the nozzle is at $x = 0.30$ m, where it is exposed to back pressure $P_b = 50.0$ kPa. In the tank,

- $P_{0,\text{inlet}} = 220$ kPa (absolute)
- $T_{0,\text{inlet}} = 300$ K

The cross-sectional area A is given as a function of axial distance x (in an Excel spreadsheet).



To do:

(a) For isentropic flow through the nozzle, calculate and plot Mach number M and pressure P (in kPa) as functions of nozzle axial distance x.

(b) Repeat for P_b =180. kPa.

Solution:

Assumptions and Approximations:

1. The air is an ideal gas with $\gamma = 1.4$.
2. The flow is steady and can be approximated as isentropic, adiabatic, and one-D.

Some equations we may need:

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \qquad \frac{T}{T_0} = \left(1+\frac{\gamma-1}{2}M^2\right)^{-1} \qquad \frac{P}{P_0} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{-\gamma}{\gamma-1}}$$

$$\frac{\rho}{\rho_0} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{-1}{\gamma-1}}$$

calc M @ each x (iteratively since implicit)

THEN FIND P

decide if choked or not

(**a**) We calculate $P_b/P_{0,\text{inlet}} = 50/220 = 0.2273$ –this back pressure is low enough (less than 0.5283 for air) that **the flow is choked**. **Therefore, the exit area *is* A^*.** ☆

For each x, we plug in the given area A at that x and calculate M implicitly. M increases from 0 (stagnation/no flow in tank) to 1 at the exit plane and P decreases from P_0 to $\mathbf{P^* = 116.22}$ **kPa** at the exit plane.

- Since flow is choked, $A^* = A_e$
- Use $\frac{A}{A^*}$ vs M eq. to get M as a fnc of x

Caution → 2 roots

- Subsonic root ← choose this one
- Supersonic root

- Now that we have M, calc P

$$\boxed{\boxed{P = \frac{P}{P_0} P_0} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma}{\gamma - 1}} P_0}$$

- At exit plane, we know it is choked, ∴ $A_e = A^*$

"e" = exit plane

$M_e = 1$

$P_e = P^*$

recall, $\boxed{\frac{P^*}{P_0} = \left(\frac{\gamma + 1}{2}\right)^{-\frac{\gamma}{\gamma - 1}}} \rightarrow \boxed{P_e = P^* = 116.22 \text{ kPa}}$

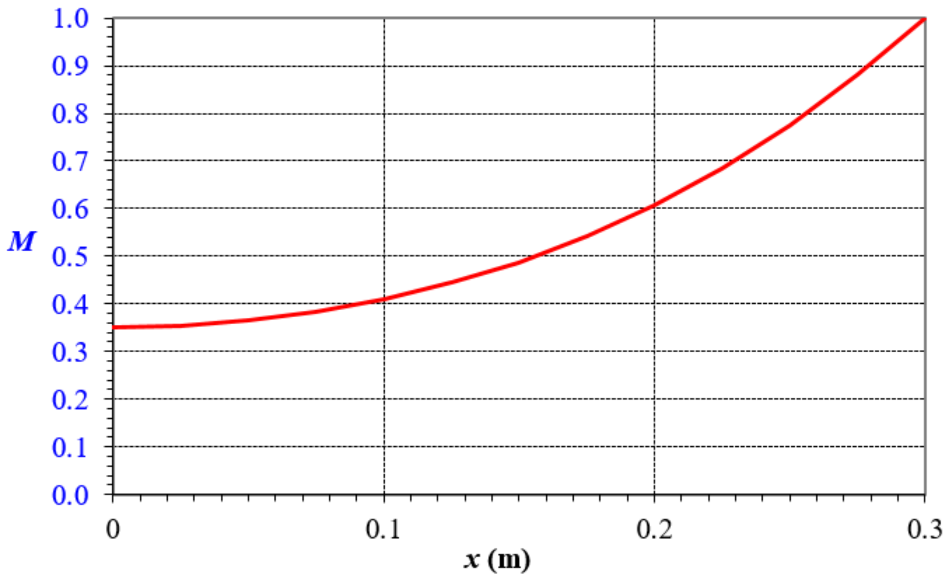
See Excel screenshot →

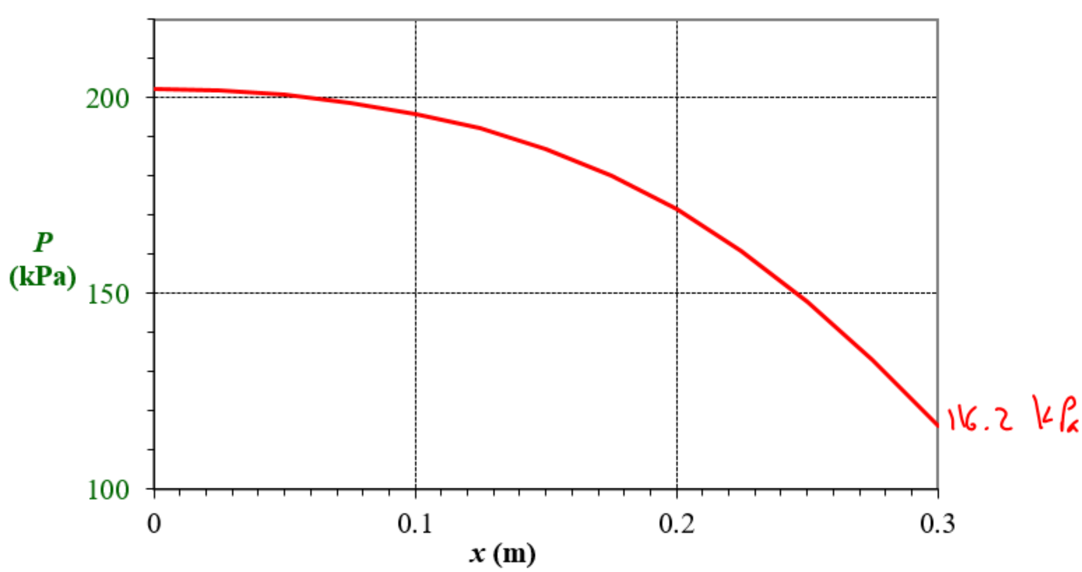
Results from Excel (*choked case*):

I used Newton's method (horizontally for each row)

	x (m)	A (m²)	A/A^*	Final M	P (kPa)
inlet	0	0.031415927	1.777777778	0.350044	202.1258
	0.025	0.031151243	1.762799754	0.353525	201.7879
	0.05	0.0303351	1.716615507	0.364752	200.6805
	0.075	0.029093379	1.646348473	0.383436	198.7794
	0.1	0.02755741	1.559430407	0.409753	195.9833
	0.125	0.025855012	1.463094387	0.444243	192.1235
	0.15	0.02410492	1.364059439	0.48773	186.9705
	0.175	0.022413343	1.268335762	0.541265	180.248
	0.2	0.02087258	1.181146411	0.606052	171.6613
	0.225	0.019561706	1.106966117	0.683376	160.9515
	0.25	0.01854931	1.049676223	0.774494	147.98
	0.275	0.01789831	1.012837159	0.880463	132.8401
exit	0.3	0.017671459	1	1	116.222

$A^* = A_e$ $M_e = 1$ ☺ This $= p^*$ ☺





(b) Repeat for P_b =180. kPa. $> P^*$

This flow is ***not choked*** (remains subsonic everywhere in the converging nozzle) since $P_b > P^*$ (P^* = 116.22 kPa).

$P_e = 180$ kPa [SINCE NOT CHOKED]

- Subsonic everywhere → $$\frac{P_e}{P_0} = \left[1 + \frac{\gamma-1}{2} M_e^2\right]^{\frac{-\gamma}{\gamma-1}}$$

Solve for M_e

$$\frac{P_e}{P_0} = \frac{P_b}{P_0} = \frac{180.\ \text{kPa}}{220\ \text{kPa}} = 0.818181\ldots$$

$$M_e = 0.54318$$

$$\boxed{M_e = 0.543}$$

Now calc M & P for each x (i.e., for each $\frac{A}{A^*}$)

Here $A_e \neq A^*$ since not choked!

But we need A^* to calc M @ each x

Q — What is A^* for this case?

$$\boxed{\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

Applies @ exit plane where we know $M = M_e = 0.54318$

$$\frac{A_e}{A^*} = \frac{1}{M_e}\left[\left(\frac{2}{\gamma+1}\right)\left(1 + \frac{\gamma-1}{2}M_e^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

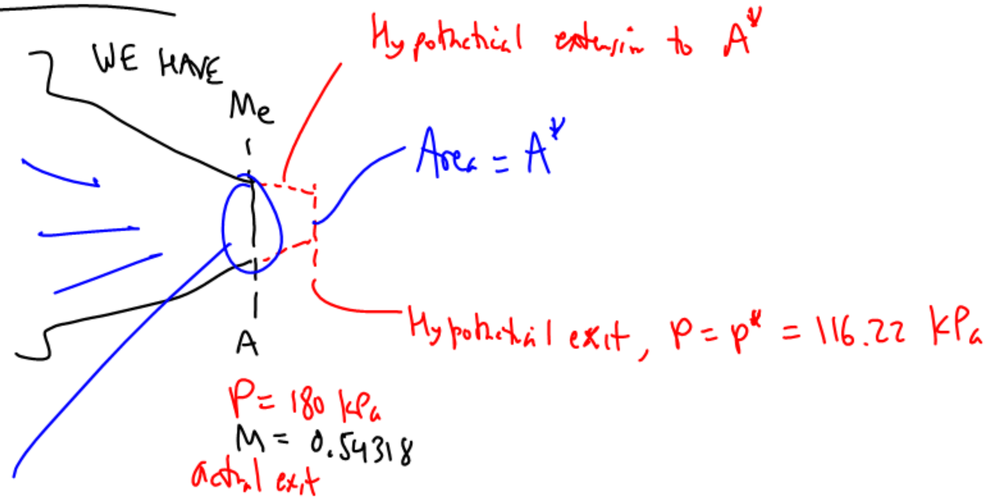
— Apply @ M_e, A_e & solve for A^* (explicit)

$$\frac{A_e}{A^*} = 1.2653 \rightarrow A^* = \frac{A_e}{(A_e/A^*)} = \boxed{0.13952\ m^2 = A^*}$$

THIS A^* DOES NOT EXIST IN THIS FLOW!

notice $A^* < A_e$

Physical explanation



THIS REAL EXIT PLANE HAS SAME PROPERTIES FOR REAL & IMAGINARY CASES

Use this A^* (even though it does not exist)

to calc M & then P @ every x location

I REPEATED (copy & paste) with new A^*

In Excel

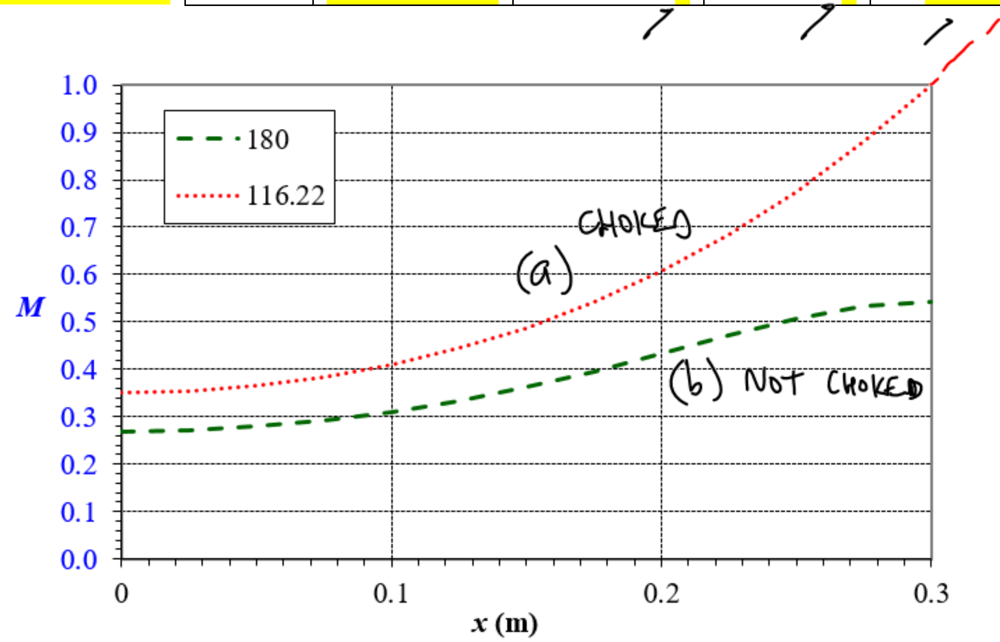
Results from Excel (*non-choked case*): (b)

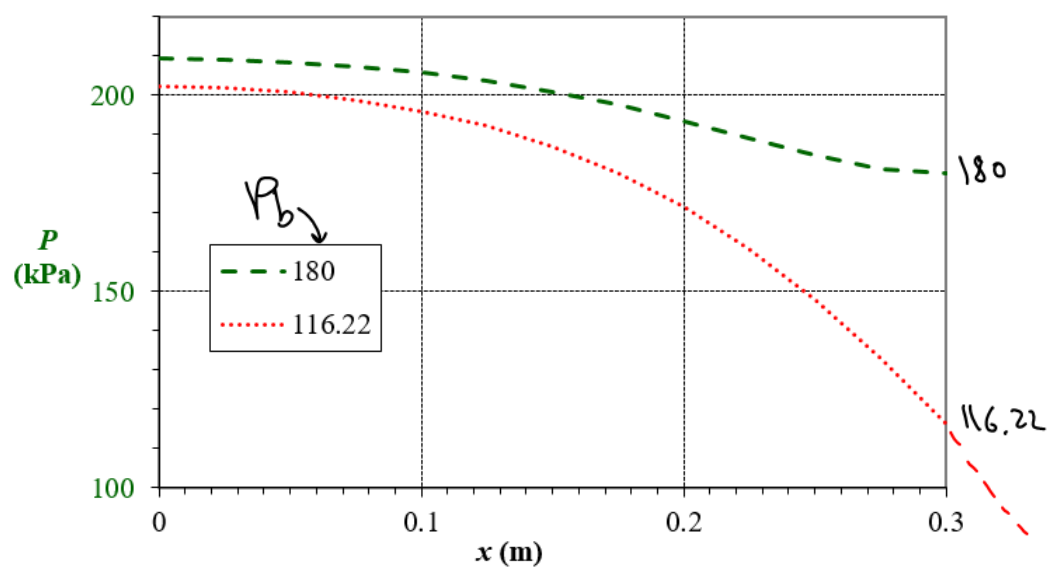
	x (m)	A (m^2)	A/A^*	Final M	P (kPa)
inlet	0	0.031415927	2.249500913	0.268552	209.2446
	0.025	0.031151243	2.230548556	0.27105	209.0501
	0.05	0.0303351	2.172109585	0.279069	208.4152
	0.075	0.029093379	2.083197595	0.292279	207.3345
	0.1	0.02755741	1.97321632	0.310583	205.7676
	0.125	0.025855012	1.85131809	0.333981	203.6506
	0.15	0.02410492	1.726004786	0.362407	200.9141
	0.175	0.022413343	1.60488138	0.395502	197.5141
	0.2	0.02087258	1.494556836	0.432259	193.4886
	0.225	0.019561706	1.400693227	0.470496	189.0485
	0.25	0.01854931	1.328201787	0.506176	184.698
	0.275	0.01789831	1.281587689	0.532966	181.3146
exit	0.3	0.017671459	1.265344263	0.543185	180
Hypothetical extension		0.013965732	1	1	116.222

real exit plane

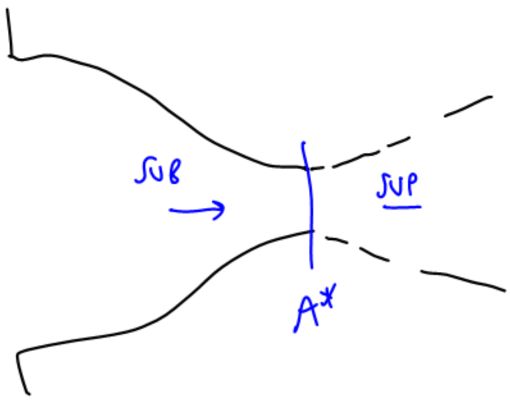
Actual nozzle

COMPARE PART (a) : (b)

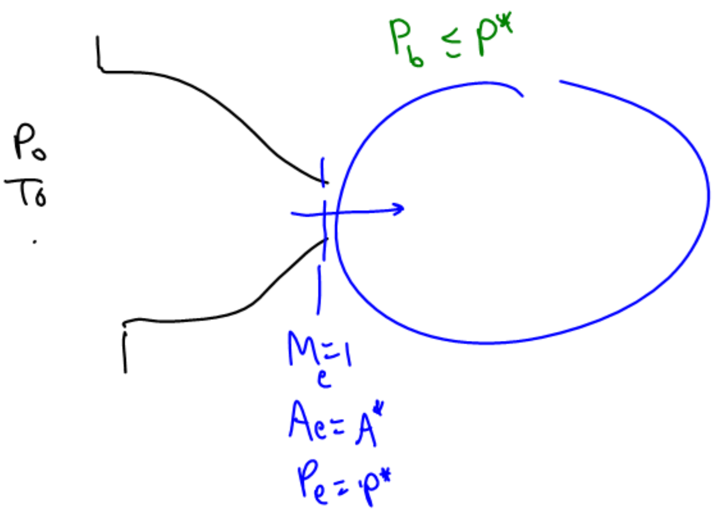




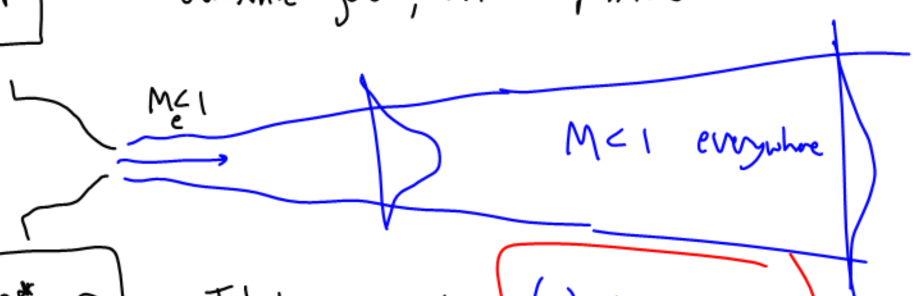
LATER ...



WHAT HAPPENS DOWNSTREAM OF A CHOKED NOZZLE ?



• If $\boxed{P_b > p^*}$ → subsonic jet, but compressible

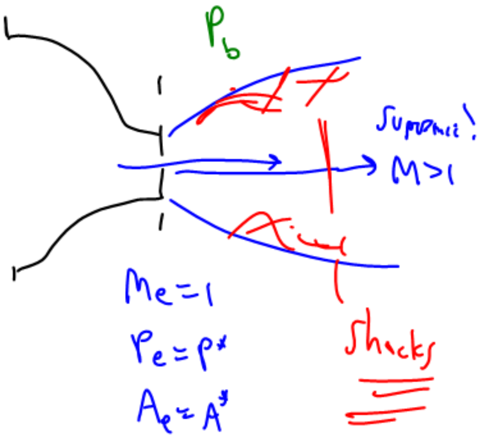


• If $\boxed{P_b = p^* = P_e}$ Ideal sonic jet (Ideal expansion)

• $\boxed{p_b < p^*}$ • $M = 1$ @ exit plane since choked

$p_e = p^*$

But now, $\underline{p_b < p_e = p^*}$

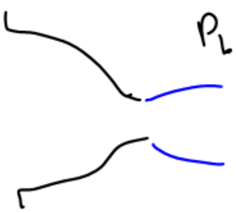


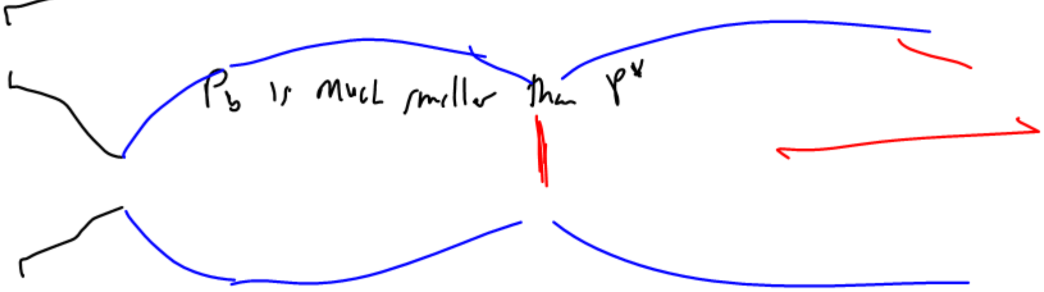
UNDEREXPANDED JET ✱

JET EXPANDS OUTWARD

As $p_b \downarrow$, expansion angle increases

p_b is a little smaller than p^*





Drawing of an underexpanded sonic jet:

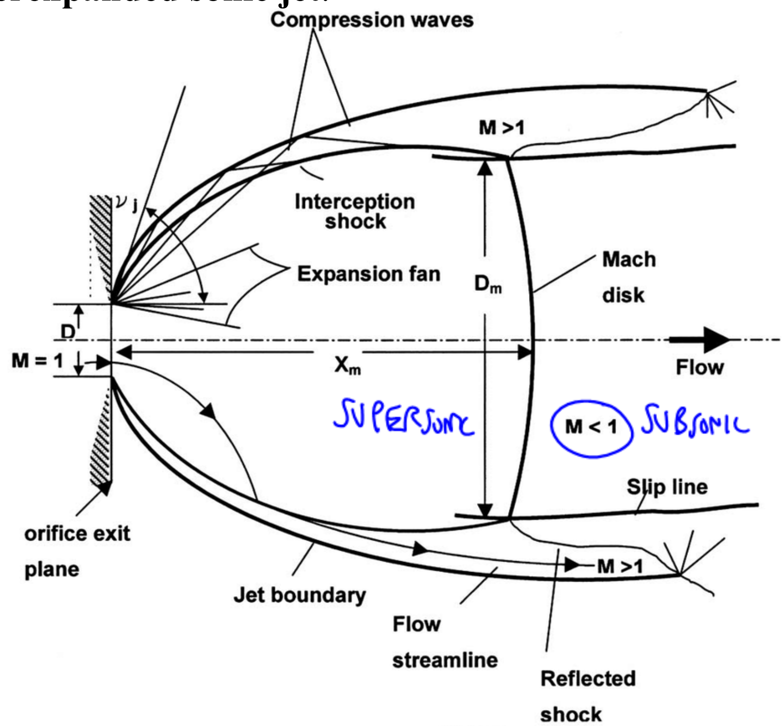


Image from Apurva Bhagat, Harshal Gijare, and Nishanth Dongari, https://www.researchgate.net/publication/322330976_NUMERICAL_INVESTIGATION_OF_MULTI-SPECIES_UNDER-EXPANDED_SONIC_JETS

Photo of an underexpanded sonic jet:

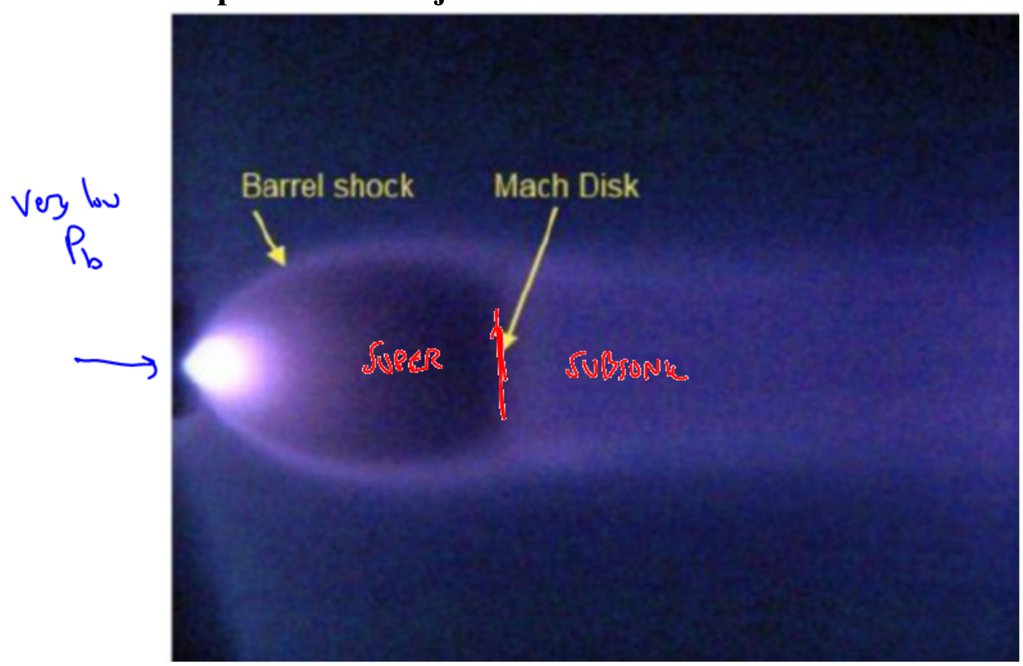


Image from: Uzi Even, "The Even-Lavie valve as a source for high intensity supersonic beam," December 2015, DOI: 10.1140/epjti/s40485-015-0027-5. Figure caption from the article: CW gas jet expanding from sonic nozzle into poor vacuum (from [60]) showing the shock wave structure in jets. Barrel shock wave, Mach disk and the "Zone of silence [7]" between them are clearly visible in this rare photograph. The gas was made to glow by electron beam excitation.

Farther downstream:

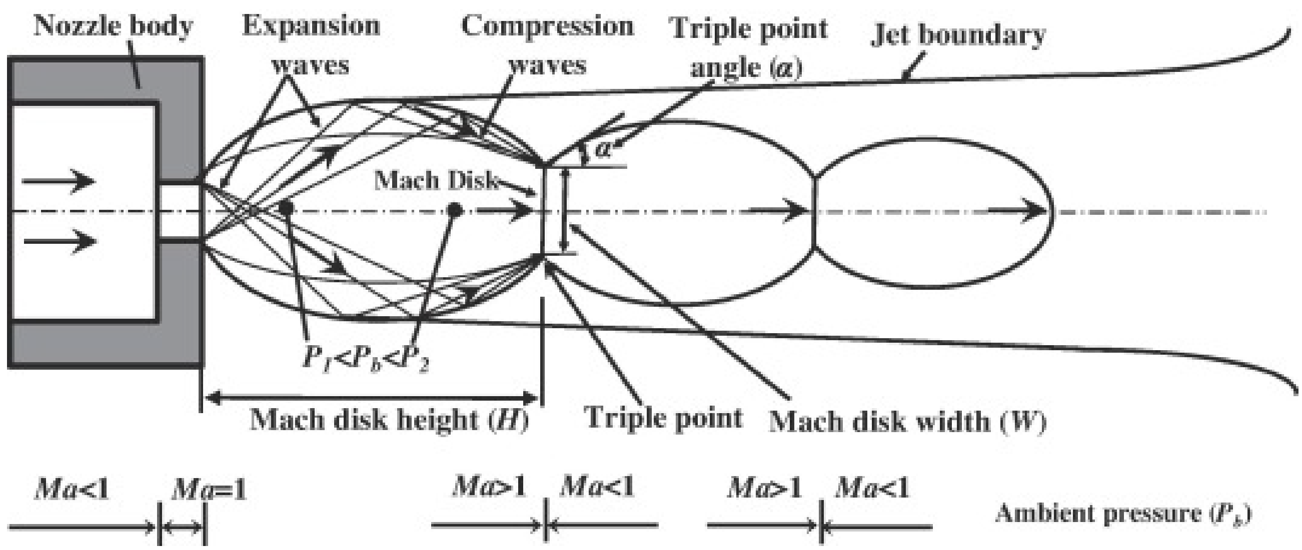


Image from Quan Dong, Yue Li, Enzhe Song, Chong Yao, Liyun Fan, and Jun Sun, *Energy Conversion and Management*, Volume **149**, 1 October 2017. https://www.sciencedirect.com/science/article/pii/S0196890417305587 .

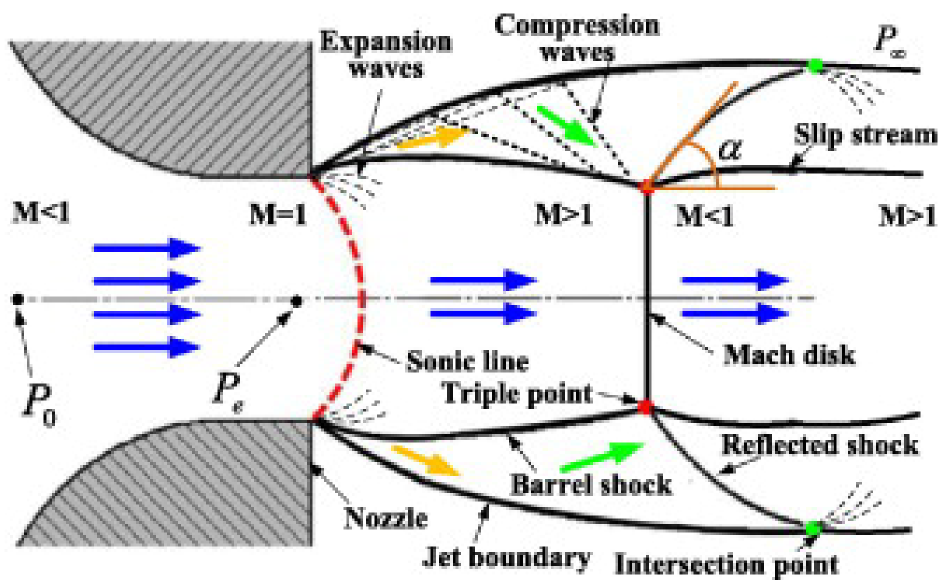
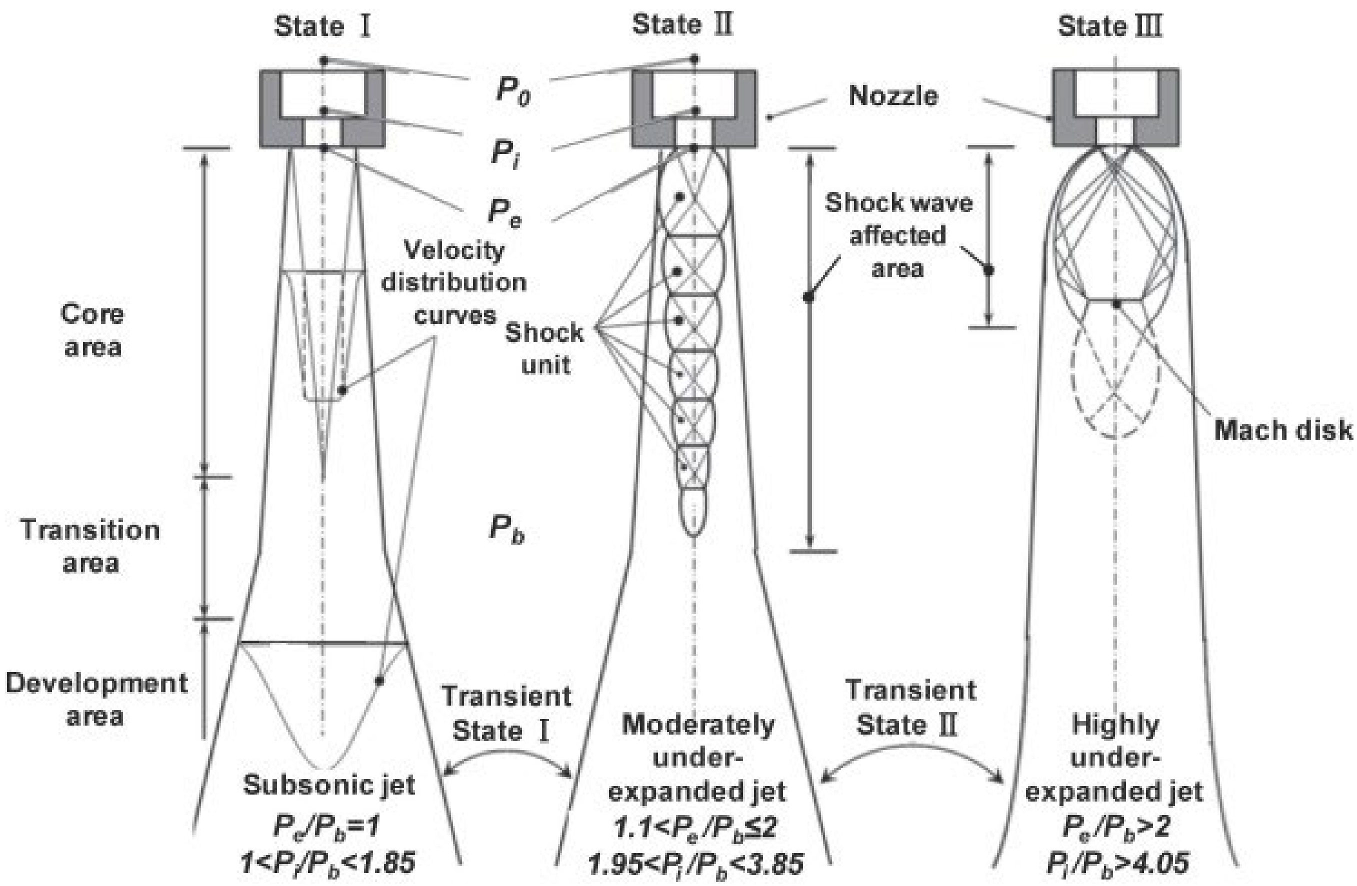


Image from Jingzhou Yu, Ville Vuorinen, Ossi Kaario, Teemu Sarjovaara. And Martti Larmi, *International Journal of Heat and Fluid Flow*, Volume **44**, December 2013. https://www.sciencedirect.com/science/article/pii/S0142727X13001227 .

Comparison of subsonic, moderately under-expanded, and highly under-expanded jets:



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