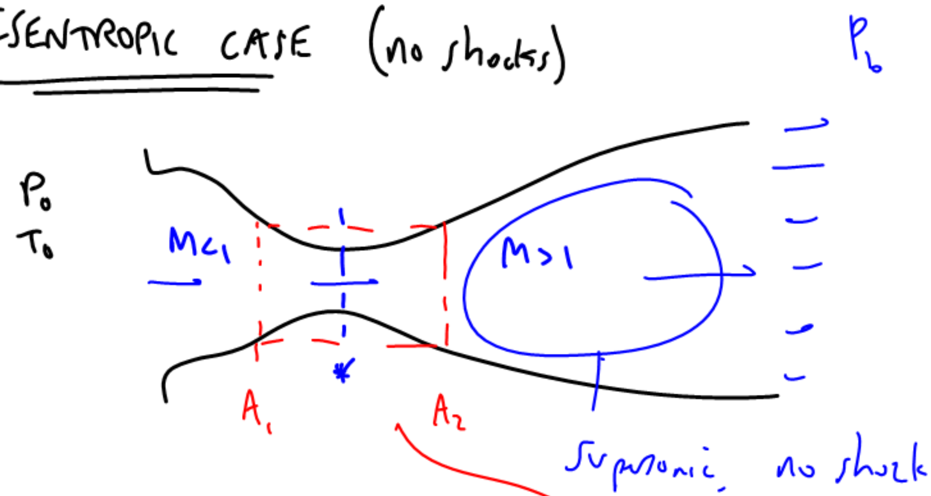
Today, we will:

- Do a ***quantitative*** analysis (equations) for C-D nozzles without shocks
- Do some example problems – flow in C-D nozzles without shocks

☆ ISENTROPIC CASE (no shocks)



$A_1 = A_2 \rightarrow \frac{A}{A^*} = $ ~

$$\boxed{\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}} \quad ☆$$

2 roots

Pick subsonic root in converging part
Pick supersonic root in diverging part

Do an example

Example: Converging ~~nozzle~~ - Diverging Nozzle

Given: Air flows from a very large tank through a converging-diverging nozzle. The back pressure is low enough that the flow is choked throughout the entire nozzle (no shocks – condition E, F, or G in previous discussion). The throat area is 0.015 m^2.

To do: Calculate the Mach number upstream and downstream of the throat at the two locations where A = 0.020 m^2.

Solution:

Assumptions and Approximations:

1. The air is an ideal gas with γ = 1.4.
2. The flow is steady and can be approximated as isentropic, adiabatic, and one-D.

$$\frac{A}{A^*} = \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

To be completed in class.

Solve implicitly

$$\frac{A}{A^*} = \frac{0.020 \text{ m}^2}{0.015 \text{ m}^2} = 1.3333\ldots$$

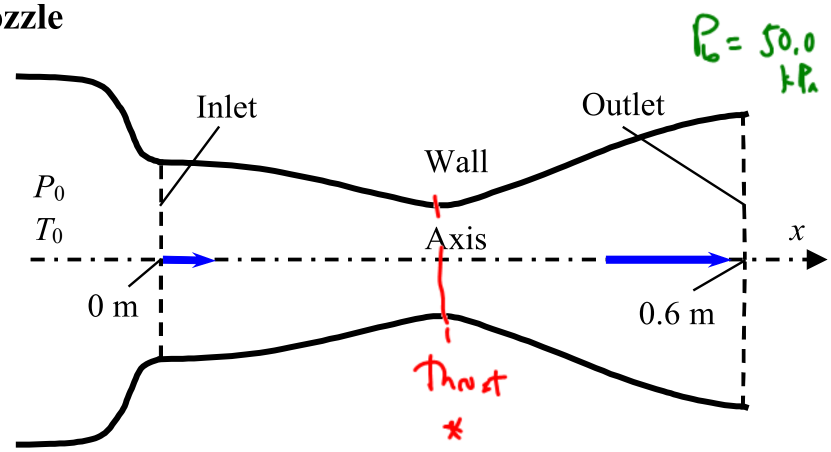
2 roots → $\boxed{M = 0.5034 \quad M = 1.695}$

Example: Converging-Diverging nozzle

Given: Air flows from a very large tank through a converging-diverging nozzle. The test section begins at $x = 0$. The outlet of the test section is at $x = 0.60$ m, where it is exposed to back pressure $P_b = 50.0$ kPa. In the tank,

- $P_{0,\text{inlet}} = 220$ kPa (absolute)
- $T_{0,\text{inlet}} = 300$ K

The cross-sectional area is known as a function of axial distance x:



	x (m)	A (m^2)	
inlet	0	0.031415927	
	0.025	0.031151243	
	0.05	0.0303351	
	0.075	0.029093379	
	0.1	0.02755741	
	0.125	0.025855012	
	0.15	0.02410492	
	0.175	0.022413343	
M<1	0.2	0.02087258	
	0.225	0.019561706	
	0.25	0.01854931	
	0.275	0.01789831	
throat	0.3	0.017671459	← $= A^*$
	0.325	0.01807656	
	0.35	0.01925422	
	0.375	0.021124432	
→	0.4	0.02360668	← Pick this row for example calcs
	0.425	0.026600575	
M>1	0.45	0.02997014	
	0.475	0.033534303	
	0.5	0.03706572	
	0.525	0.040297166	
	0.55	0.04293412	
	0.575	0.044674717	
outlet	0.6	0.045238934	

(a) To do: For isentropic flow through the converging-diverging nozzle (no shocks), calculate and plot Mach number and nondimensional pressure $P/P_{0,\text{inlet}}$ as functions of x.

Solution:

Assumptions and Approximations:

1. The air is an ideal gas with $\gamma = 1.4$. The flow is steady, one-D, adiabatic, & isentropic.
2. We calculate $P_b/P_{0,\text{inlet}} = 50/220 = 0.2273$ – we *assume* this back pressure is low enough that the flow is supersonic through the entire diverging section of the nozzle, without any normal shocks in the nozzle. *We need to verify this assumption later.*

Some equations we may need:

$$\frac{A}{A^*}=\frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{T}{T_0}=\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}$$

$$\frac{P}{P_0}=\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{-\gamma}{\gamma-1}}$$

$$\frac{\rho}{\rho_0}=\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{-1}{\gamma-1}}$$

ISENTROPIC EQS

To be completed in class.

SAMPLE CASE @ $x=0.40$ m

here, $\frac{A}{A^*}=\frac{0.0236067\ m^2}{0.0176715\ m^2}=1.33587$

From $\frac{A}{A^*}$ vs M eq $\rightarrow$ implicitly solve $\rightarrow$ $\boxed{M=1.69815}$

(Supersonic root)

$$\frac{P}{P_0}=\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{-\gamma}{\gamma-1}} \rightarrow \boxed{\frac{P}{P_0}=0.20316}$$

can calc $P=\left(\frac{P}{P_0}\right)P_0$

$P=44.7$ kPa @ $x=0.40$ m

Do similar calcs for $\frac{\rho}{\rho_0}$, $\frac{T}{T_0}$, $\rightarrow \rho, T$ vs x

Repeat @ all x locations

Final tabulated results (I used Excel) and plot:

				CFD Results:		Isentropic Flow Calculations (Theory):	
	x (m)	*A* (m^2)	*A*/*A**	*M*	P/P_0	Final *M*	P/P_0
inlet	0	0.031415927	1.777778	0.350029	0.918791	0.350044	0.918754
	0.025	0.031151243	1.7628	0.353946	0.917069	0.353525	0.917218
	0.05	0.0303351	1.716616	0.365743	0.911777	0.364752	0.912184
	0.075	0.029093379	1.646348	0.385	0.902856	0.383436	0.903543
	0.1	0.02755741	1.55943	0.411804	0.889879	0.409753	0.890833
	0.125	0.025855012	1.463094	0.446605	0.872126	0.444243	0.873288
	0.15	0.02410492	1.364059	0.490141	0.848617	0.48773	0.849866
	0.175	0.022413343	1.268336	0.543382	0.818169	0.541265	0.819309
	0.2	0.02087258	1.181146	0.607469	0.779519	0.606052	0.780279
	0.225	0.019561706	1.106966	0.683631	0.731561	0.683376	0.731598
	0.25	0.01854931	1.049676	0.773087	0.673691	0.774494	0.672636
	0.275	0.01789831	1.012837	0.876967	0.606227	0.880463	0.603819
throat	0.3	0.017671459	1	0.997128	0.530239	1	0.528282
	0.325	0.01807656	1.022924	1.172304	0.427974	1.172677	0.427242
	0.35	0.01925422	1.089566	1.346716	0.339248	1.351084	0.336465
	0.375	0.021124432	1.195398	1.518473	0.265941	1.527721	0.261647
	0.4	0.02360668	1.335865	1.683724	0.208268	1.698149	0.20316
	0.425	0.026600575	1.505285	1.839401	0.164328	1.858632	0.159069
	0.45	0.02997014	1.695963	1.996483	0.128781	2.005996	0.126618
	0.475	0.033534303	1.897653	2.124023	0.10542	2.137473	0.103132
	0.5	0.03706572	2.097491	2.235109	0.088518	2.250561	0.086406
	0.525	0.040297166	2.280353	2.328241	0.076595	2.342862	0.074787
	0.55	0.04293412	2.429574	2.401027	0.068704	2.411878	0.067143
	0.575	0.044674717	2.528072	2.451325	0.06433	2.454801	0.062795
outlet	0.6	0.045238934	2.56	2.477948	0.063394	2.468307	0.061487

For CFD, these are averaged across the cross-sectional area

I also ran a CFD simulation of this flow (circles), and compared with theory (lines).

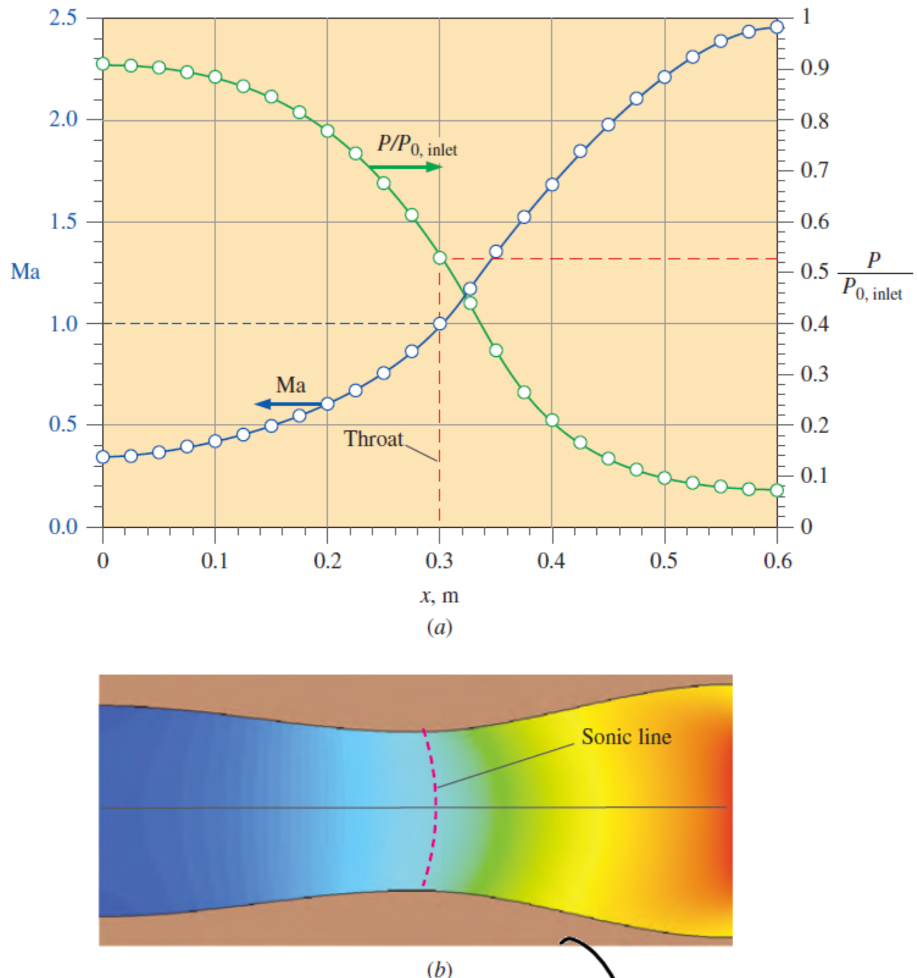


FIGURE 15–76
CFD results for steady, adiabatic, inviscid compressible flow through an axisymmetric converging–diverging nozzle: (*a*) calculated average Mach number and pressure ratio at 25 axial locations (circles), compared to predictions from isentropic, one-dimensional compressible flow theory (solid lines); (*b*) Mach number contours, ranging from Ma = 0.3 (blue) to 2.7 (red). Although only the top half is calculated, a mirror image about the *x*-axis is shown for clarity. The sonic line (Ma = 1) is also highlighted. It is parabolic rather than straight in this axisymmetric flow due to the radial component of velocity, as discussed in Schreier (1982).

Figure 15-76 from Cengel & Cimbala

- Agreement is excellent! (CFD vs. theory)
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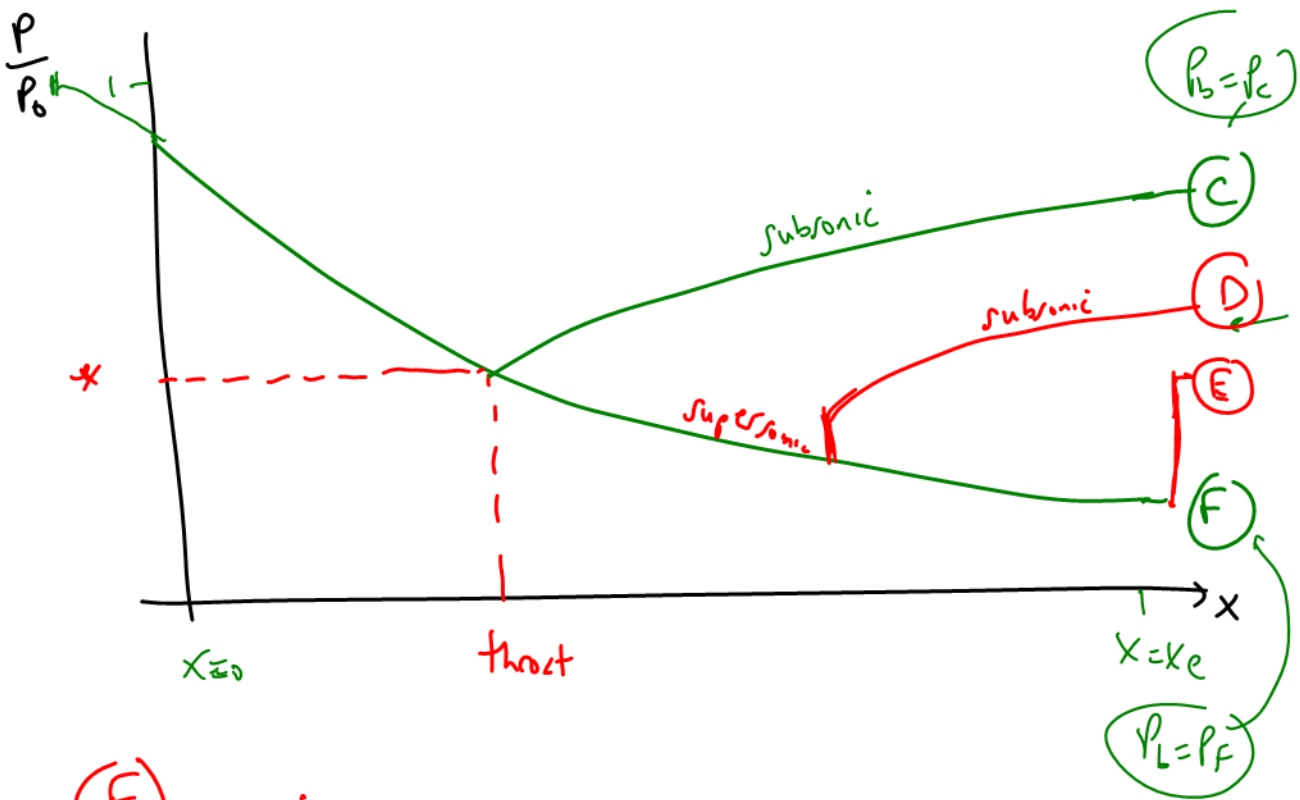
ISO - Mach # contour plot

- CFD requires
  - mass
  - moment (N-S eq.)
  - energy eq.

; $\underline{\text{Eq of state}}$ (ideal gas)

* VERIFY — How can we be sure there are no shocks?

P_b must be "small enough"



(E) = condition where shock @ EXIT PLANE

For my assumption to be valid (no shock in nozzle),

P_b must be $< P_E$

At exit plane (x = 0.60 m) from table → $\boxed{\frac{P_e}{P_0} = 0.061487}$

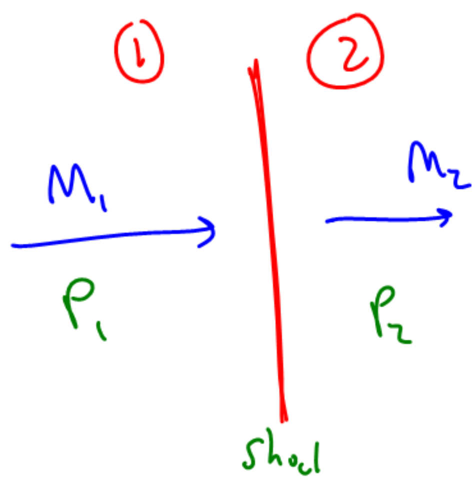
$\boxed{\text{OUR } P_b = 50.0 \text{ kPa}}$ $\boxed{P_e = 13.5271 \text{ kPa}}$

$P_F = P_e$ = ideal expansion case

We need to calc P_E to answer our question

(Worst case scenario to have no shock in nozzle)

NORMAL SHOCK



$M \downarrow$
$P \uparrow$

We predicted that $P_e = 13.5271$ kPa, $M_e = \boxed{2.4683}$ } P_1 & M_1

For a shock @ exit plane (case E), let's calc. M_2 & P_2

$\boxed{P_2 \gg P_1}$ → COMP. AERO. CALCULATOR → $\boxed{\dfrac{P_2}{P_1} = 6.94126}$

$$\therefore\ P_2 = \left(\frac{P_2}{P_1}\right) \underbrace{P_1}_{P_e} = \boxed{93.89 \text{ kPa} = P_E}$$

OUR $P_b = 50.0$ kPa. Since $P_b < P_E$, our assumption is valid