Today, we will:

- Do an alternate (more elegant) derivation of M_2 across a normal shock
- Introduce the Prandtl relation for flow across a normal shock
- Generate equations for how *other* properties (P, T, P_0, ρ, V, etc.) change across a normal shock (1 to 2)

ALTERNATE DERIVATION OF M_2 ACROSS A SHOCK
(in all textbooks)

Start w/ Eq (5)

$$\boxed{P_1 - P_2 = \rho_2 V_2^2 - \rho_1 V_1^2} \qquad (5)$$

linear mom. across a shock

- $\div \rho_1 V_1$
- apply cons. of mass

$$\boxed{\rho_1 V_1 = \rho_2 V_2}$$

$$\frac{P_1}{\rho_1 V_1} - \frac{P_2}{\rho_2 V_2} = V_2 - V_1$$

- $a = \sqrt{\gamma R T} = \sqrt{\gamma \frac{P}{\rho}} \Rightarrow \boxed{P = \frac{\rho a^2}{\gamma}}$

$$\boxed{\frac{a_1^2}{\gamma V_1} - \frac{a_2^2}{\gamma V_2} = V_2 - V_1} \qquad (17)$$

(from mass & momentum only)

NOW, APPLY ENERGY EQ

recall, $$\boxed{h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}} \qquad (2)$$

$$\boxed{h + \frac{V^2}{2} = h_0 = \text{constant}}$$

TRUE
even across a shock
since adiabatic

manipulate

$$c_pT + \frac{V^2}{2} = \text{const}$$

$$\boxed{c_p = R\frac{\gamma}{\gamma-1}}$$

$$\frac{\gamma RT}{\gamma-1} + \frac{V^2}{2} = \text{const}$$

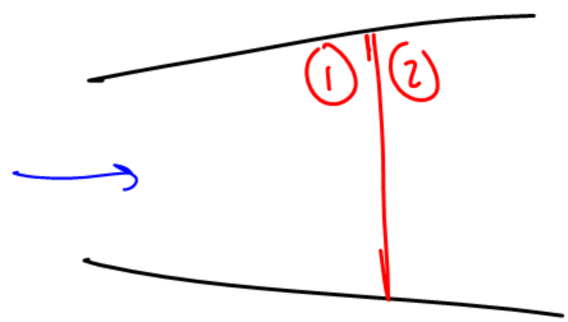
a^2

$$\boxed{\frac{a^2}{\gamma-1} + \frac{V^2}{2} = \text{const}} \qquad (18)$$

?

Define * as critical or sonic state where $M=1$

we can use * state even if * conditions do not actually exist in our flow



@* $M=1$

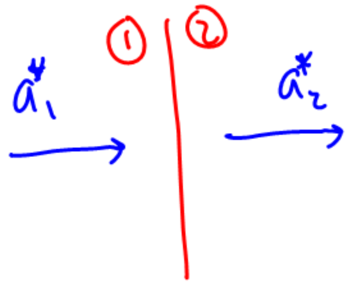
$$V = a = \underline{\underline{a^*}}$$

$$(18) \longrightarrow \quad \frac{a^2}{\gamma-1} + \frac{V^2}{2} = \frac{a^{*2}}{\gamma-1} + \frac{a^{*2}}{2}$$

$$\boxed{\frac{a^2}{\gamma-1} + \frac{V^2}{2} = \frac{\gamma+1}{2(\gamma-1)} a^{*2}} \quad \star \ (19)$$

or

$$\boxed{a^2 = \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} V^2} \quad (19a)$$



BUT $a_1^* = a_2^* = a^*$

a^* does not change across a shock

$$\left.\boxed{\begin{aligned} @① \quad a_1^2 &= \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} V_1^2 \\ @② \quad a_2^2 &= \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} V_2^2 \end{aligned}}\right\} (20)$$

PLUG (20) into (17)

$$\frac{\gamma+1}{2}\frac{a^{*2}}{\gamma V_1} - \frac{\gamma-1}{2}\frac{V_1^2}{\gamma V_1} - \frac{\gamma+1}{2}\frac{a^{*2}}{\gamma V_2} + \frac{\gamma-1}{2}\frac{V_2^2}{2\gamma V_2} = V_2 - V_1$$

combine

combine

get $\boxed{a^{*2} = V_1 V_2}$ (21)

PRANDTL RELATION FOR SHOCKS

Recall, $\boxed{M^* = \frac{V}{a^*}}$ = characteristic Mach #

$a^* = \sqrt{\gamma R T^*}$

(21) $\rightarrow$ $a^* a^* = V_1 V_2$

$$1 = \frac{V_1}{a^*}\frac{V_2}{a^*} = M_1^* M_2^* \Rightarrow \boxed{M_2^* = \frac{1}{M_1^*}} \quad (22)$$

Recall, previous lecture $\rightarrow$

$$\boxed{M^2 = \frac{2}{\frac{\gamma+1}{M^{*2}} - (\gamma-1)}} \quad (23)$$

If $\boxed{M^* = 1}$, $M^2 = \frac{2}{1+1} = 1 \rightarrow \boxed{M = 1}$

If $M^* < 1$, $M < 1$

If $M^* > 1$ $M > 1$

Relation between M & M^* @ any state

applies anywhere in the flow

Eq (23) can be solved for M^*

$$\boxed{M^{*2} = \frac{M^2(\gamma+1)}{2+(\gamma-1)M^2}} \quad (24)$$

- Plug (24) into (22)

$$M_2^* = \frac{1}{M_1^*} \Rightarrow \underbrace{\frac{M_2^2(\gamma+1)}{2+(\gamma-1)M_2^2} = \frac{2+(\gamma-1)M_1^2}{M_1^2(\gamma+1)}}$$

Solve for $M_2 = \text{fnc}(M_1, \gamma)$

$$\boxed{M_2^2 = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}} \quad (25)$$

Second eq. for $M_2 = \text{fn}(M_1, \gamma)$ across a shock

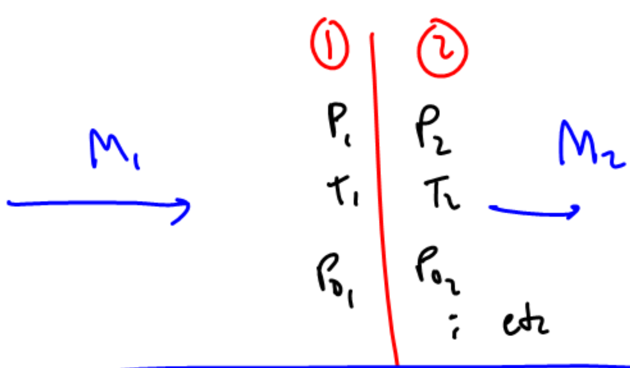
$$\boxed{M_2 = \sqrt{\frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}}} \quad \star \qquad (25)$$

E.g., same M_1 as previously; we solved for M_2 for air: $\boxed{\gamma = 1.40}$; $\boxed{M_1 = 2.60}$

- Website calculator → $\underline{M_2 = 0.503871}$
- my quadratic rule → "

Plug in here, $\underline{\underline{M_2 = 0.503871}}$ ☺

OTHER EQ ACROSS A NORMAL SHOCK



We $\boxed{M_2 = \sqrt{\frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 - \frac{\gamma-1}{2}}}}$

restart #s → (1)

$\frac{P_2}{P_1} =$ use Rayleigh or Fanno

$$\boxed{\frac{P_2}{P_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}} \qquad (2)$$

algebra: combine (1) & (2)

$$\boxed{\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - \gamma + 1}{\gamma + 1}} \quad \star \quad (3)$$

e.g.,
$$\frac{\rho_2}{\rho_1} = \frac{P_2}{R T_2}\frac{R T_1}{P_1} = \frac{P_2}{P_1} \Big/ \left(\frac{T_2}{T_1}\right)$$

recall,
$$\frac{T_2}{T_1} = \frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2} \qquad (4)$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2}{2 + (\gamma-1) M_1^2}} \quad \star \qquad (5)$$

Can do similar manipulation to get $\frac{T_2}{T_1} = \text{fnc}(M_1, \gamma)$

Cons. of mass → $\rho_1 V_1 = \rho_2 V_2$

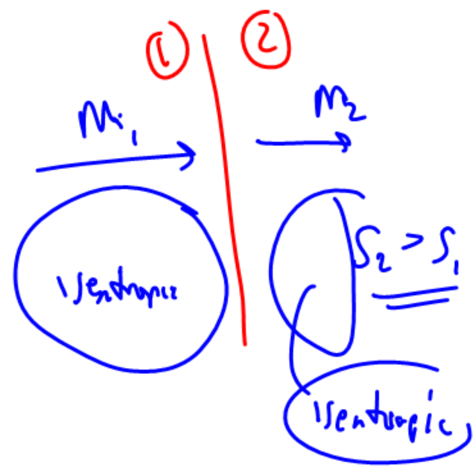
$$\therefore \boxed{\frac{V_2}{V_1} = \frac{\rho_1}{\rho_2}} \quad ✪$$

$$\therefore \boxed{\frac{V_2}{V_1} = \left[\frac{\gamma+1\, M_1^2}{2+(\gamma-1)M_1^2}\right]^{-1}} \qquad (6)$$

✪ **Stagnation Properties across a shock**

adiabatic → $\boxed{T_{0_2} = T_{0_1}}$ ✪ (7)

$\dfrac{P_{0_2}}{P_{0_1}}$?



$$\boxed{\frac{P_{0_2}}{P_{0_1}} = \frac{P_{0_2}}{P_2}\,\frac{P_1}{P_{0_1}}\,\frac{P_2}{P_1}}$$

Known from isentropic relations upstream & downstream of a shock

we just obtained this across a shock

- can use Fanno
- " " Rayleigh
- can use Eq (3)

$$\frac{p_{o_2}}{p_2} = \left[1 + \frac{\gamma-1}{2} M_2^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{p_{o_1}}{p_1} = \left[1 + \frac{\gamma-1}{2} M_1^2\right]^{\frac{\gamma}{\gamma-1}}$$

Fanno → 

$$\frac{p_{o_2}}{p_{o_1}} = \left(\frac{1 + \frac{\gamma-1}{2} M_2^2}{1 + \frac{\gamma-1}{2} M_1^2}\right)^{\frac{\gamma}{\gamma-1}} \quad \frac{M_1}{M_2}\left[\frac{1 + \frac{\gamma-1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_2^2}\right]^{\frac{1}{2}}$$

combine exponents

$$\boxed{\frac{p_{o_2}}{p_{o_1}} = \frac{M_1}{M_2}\left[\frac{1 + \frac{\gamma-1}{2} M_2^2}{1 + \frac{\gamma-1}{2} M_1^2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}} \quad ☆ \quad (8)$$

Can also plug in $M_2 = \text{fnc}(M_1, \gamma)$ eq. to get

$$\frac{p_{o_2}}{p_{o_1}} = \text{fnc}(M_1, \gamma)$$

$$\boxed{\frac{P_{0_2}}{P_{0_1}} = \left[\frac{\frac{\gamma+1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{1-\gamma+2\gamma M_1^2}{\gamma+1}\right]^{\frac{-1}{\gamma-1}}}$$

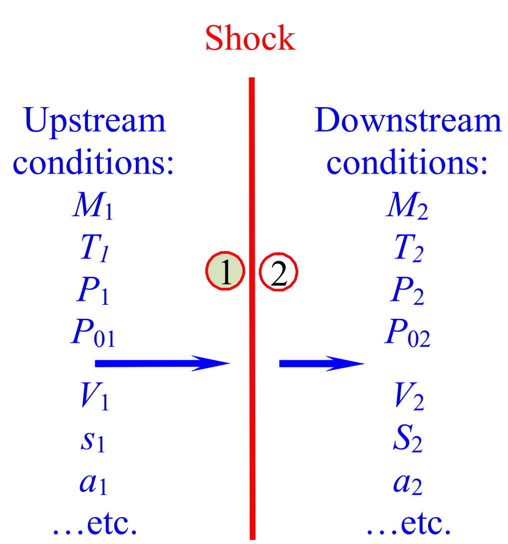
Test case: $M_1 = 2.6$ $\gamma = 1.40$ $\rightarrow$ $\boxed{\frac{P_{0_2}}{P_{0_1}} = 0.46012}$

agrees with online calculator ✓

☺

$$\boxed{P_{0_2} < P_{0_1}}$$ ☆

Summary of changes across a *stationary* normal shock wave (as in a supersonic wind tunnel)



Properties that ***increase*** across the shock:

- $P_2 > P_1$
- $T_2 > T_1$, thus:
  - $a_2 > a_1$
  - $h_2 > h_1$
- $\rho_2 > \rho_1$
- $s_2 > s_1$
- $A_2^* > A_1^*$

Properties that ***decrease*** across the shock:

- $M_2 < M_1$
- $P_{02} < P_{01}$
- $\rho_{02} < \rho_{01}$
- $V_2 < V_1$

Properties that ***stay the same*** across the shock:

- $T_{02} = T_{01}$
- $h_{02} = h_{01}$
- $a_2^* = a_1^*$

Summary of Normal Shock Equations (1 = upstream, 2 = downstream of stationary shock):

Equations and table from Cengel & Cimbala.
Note the notation differences: C&C use k instead of γ and Ma instead of M.

$$T_{01} = T_{02}$$

$$\mathrm{Ma}_2 = \sqrt{\frac{(k-1)\mathrm{Ma}_1^2 + 2}{2k\mathrm{Ma}_1^2 - k + 1}}$$

$$\frac{P_2}{P_1} = \frac{1 + k\mathrm{Ma}_1^2}{1 + k\mathrm{Ma}_2^2} = \frac{2k\mathrm{Ma}_1^2 - k + 1}{k + 1}$$

$$\frac{\rho_2}{\rho_1} = \frac{P_2/P_1}{T_2/T_1} = \frac{(k+1)\mathrm{Ma}_1^2}{2 + (k-1)\mathrm{Ma}_1^2} = \frac{V_1}{V_2}$$

$$\frac{T_2}{T_1} = \frac{2 + \mathrm{Ma}_1^2(k-1)}{2 + \mathrm{Ma}_2^2(k-1)}$$

$$\frac{P_{02}}{P_{01}} = \frac{\mathrm{Ma}_1}{\mathrm{Ma}_2}\left[\frac{1 + \mathrm{Ma}_2^2(k-1)/2}{1 + \mathrm{Ma}_1^2(k-1)/2}\right]^{(k+1)/[2(k-1)]}$$

$$\frac{P_{02}}{P_1} = \frac{(1 + k\mathrm{Ma}_1^2)[1 + \mathrm{Ma}_2^2(k-1)/2]^{k/(k-1)}}{1 + k\mathrm{Ma}_2^2}$$

TABLE A–14

One-dimensional normal shock functions for an ideal gas with $k = 1.4$

Ma_1	Ma_2	P_2/P_1	ρ_2/ρ_1	T_2/T_1	P_{02}/P_{01}	P_{02}/P_1
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.8929
1.1	0.9118	1.2450	1.1691	1.0649	0.9989	2.1328
1.2	0.8422	1.5133	1.3416	1.1280	0.9928	2.4075
1.3	0.7860	1.8050	1.5157	1.1909	0.9794	2.7136
1.4	0.7397	2.1200	1.6897	1.2547	0.9582	3.0492
1.5	0.7011	2.4583	1.8621	1.3202	0.9298	3.4133
1.6	0.6684	2.8200	2.0317	1.3880	0.8952	3.8050
1.7	0.6405	3.2050	2.1977	1.4583	0.8557	4.2238
1.8	0.6165	3.6133	2.3592	1.5316	0.8127	4.6695
1.9	0.5956	4.0450	2.5157	1.6079	0.7674	5.1418
2.0	0.5774	4.5000	2.6667	1.6875	0.7209	5.6404
2.1	0.5613	4.9783	2.8119	1.7705	0.6742	6.1654
2.2	0.5471	5.4800	2.9512	1.8569	0.6281	6.7165
2.3	0.5344	6.0050	3.0845	1.9468	0.5833	7.2937
2.4	0.5231	6.5533	3.2119	2.0403	0.5401	7.8969
2.5	0.5130	7.1250	3.3333	2.1375	0.4990	8.5261
2.6	0.5039	7.7200	3.4490	2.2383	0.4601	9.1813
2.7	0.4956	8.3383	3.5590	2.3429	0.4236	9.8624
2.8	0.4882	8.9800	3.6636	2.4512	0.3895	10.5694
2.9	0.4814	9.6450	3.7629	2.5632	0.3577	11.3022
3.0	0.4752	10.3333	3.8571	2.6790	0.3283	12.0610
4.0	0.4350	18.5000	4.5714	4.0469	0.1388	21.0681
5.0	0.4152	29.000	5.0000	5.8000	0.0617	32.6335
∞	0.3780	∞	6.0000	∞	0	∞

←For air (γ = 1.40).

Note: This table also available on the course website under the Links/Refs tab.