ME 420 Professor John M. Cimbala **Lecture 39**

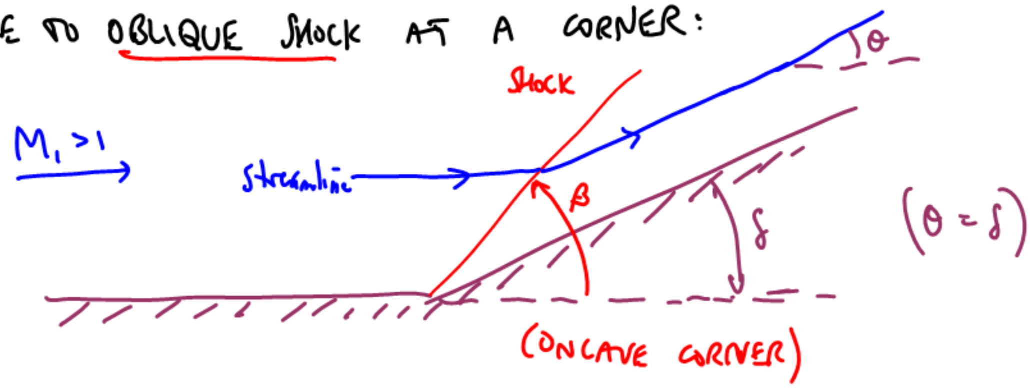
Today, we will:

- Introduce Prandtl-Meyer (P-M) expansion fans (also called P-M expansion waves)
- Give some brief histories about Prandtl and Meyer
- Develop the procedure for solving Prandtl-Meyer (P-M) expansion fan problems
- Do some example problems – P-M expansion fans
- Do Candy Questions for Candy Friday

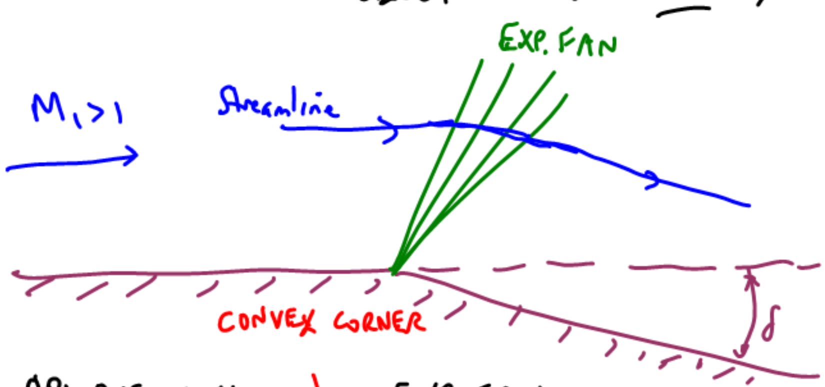
SUPERSONIC EXPANSION WAVES

[ALSO CALLED:
- PRANDTL-MEYER EXPANSION WAVES
- P-M WAVES
- EXPANSION FANS
- P-M FANS]

COMPARE TO OBLIQUE SHOCK AT A CORNER:



OPPOSITE CASE – WALL SUDDENLY DROPS DOWN, NOT UP:



THERE IS NO SUCH THING AS AN "EXPANSION SHOCK"

COMPARE:

	OBLIQUE SHOCK	EXP. FAN	
	$M_2 < M_1$	$M_2 > M_1$	OPPOSITE
$s_2 > s_1$	$T_2 > T_1$	$T_2 < T_1$	$s_2 = s_1$ (isentropic)
	$P_2 > P_1$	$P_2 < P_1$	
	$\rho_2 > \rho_1$	$\rho_2 < \rho_1$	

Historical notes about Prandtl and Meyer:

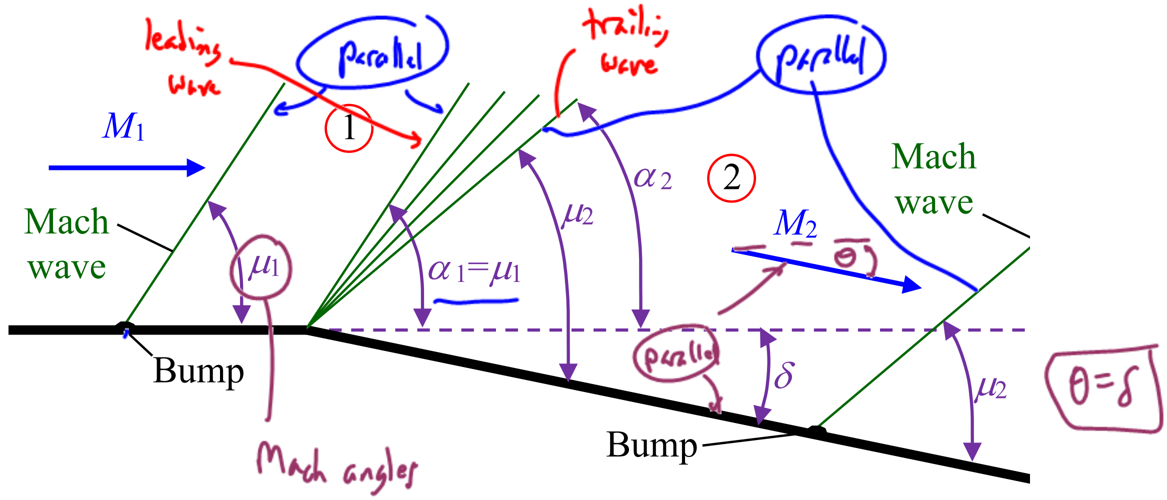
Ludwig Prandtl (1875-1953)

Theodor Meyer (1882-1971)

- **Ludwig Prandtl** studied mechanical engineering in Munich, Germany. After his PhD thesis, he married the daughter of his Ph.D. advisor, **August Föppl**.
- He started his academic career at the University of Hanover, and later went to the University of Göttingen Germany, where he served from 1904 to 1953.
- Prandtl is famous for conceiving the idea of a **boundary layer**, and his pioneering work on boundary layers.
- Prandtl's first Ph.D. student was **Heinrich Blasius**, of Blasius boundary layer fame.
- Prandtl is also famous for his pioneering work on wing theory, especially finite span wings and the concept of induced drag.
- He was the first person to estimate the thickness of a shock wave.
- He was the Ph.D. advisor for Meyer's work on oblique shocks and expansion fans.
- Prandtl invented the Pitot-static probe and contributed to turbulent mixing length theory.

- **Theodor Meyer** was another of Prandtl's early Ph.D. students at the University of Göttingen Germany, 1904-1908.
- His Ph.D. thesis (1908) presented the theory for both **oblique shock waves** and what are now known as **"Prandtl-Meyer" expansion fans**, all for the *first time*!
- Some historians claim that Meyer's thesis is *the single most important Ph.D. thesis in all of fluid dynamics*. [Others claim Blasius' thesis, also one of Prandtl's students.]
- Meyer served in the German Army in WWI, but then was unable to find a research job and spent the rest of his career teaching high-school.
- His life story was unknown until his biography was written by Settles, Krause, and Fütterer. "Theodor Meyer - Lost Pioneer of Gas Dynamics." *Progress in Aerospace Sciences* **45** (6-8):203-210, 2009.

Analysis of Prandtl-Meyer (P-M) expansion fans:



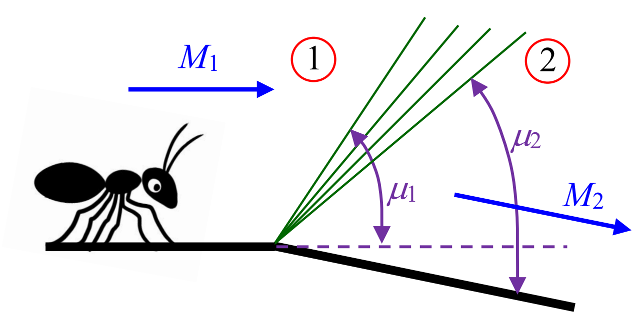
Note: Mach angle is always measured relative to the upstream flow direction.

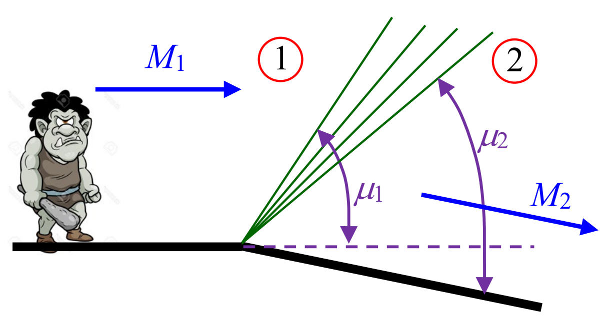
Review of what what we know so far:

- Flow through the expansion fan is approximated as ***isentropic***.
- μ_1 = Mach angle measured relative to the upstream flow, M_1. Thus, $\boxed{\mu_1 = \sin^{-1}\frac{1}{M_1}}$.

  This Mach wave is parallel to the ***leading wave*** of the expansion fan.
- μ_2 = Mach angle measured relative to the downstream flow, M_2. Thus, $\boxed{\mu_2 = \sin^{-1}\frac{1}{M_2}}$.

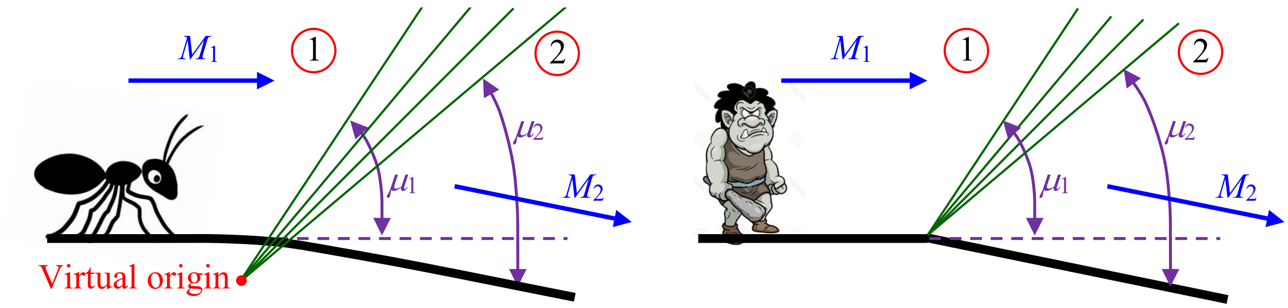
  This Mach wave is parallel to the ***trailing wave*** of the expansion fan.
- Fluid properties such as M, T, P, etc. must be ***constant along any of the P-M rays***. How do we know this? For a sharp corner, ***there is no length scale in the problem***, so we see the ***same rays and fluid properties no matter how close or far away we are looking***.





- Thus, we know that $\boxed{\mu = \sin^{-1}\frac{1}{M}}$ for ***any*** ray in the expansion fan!

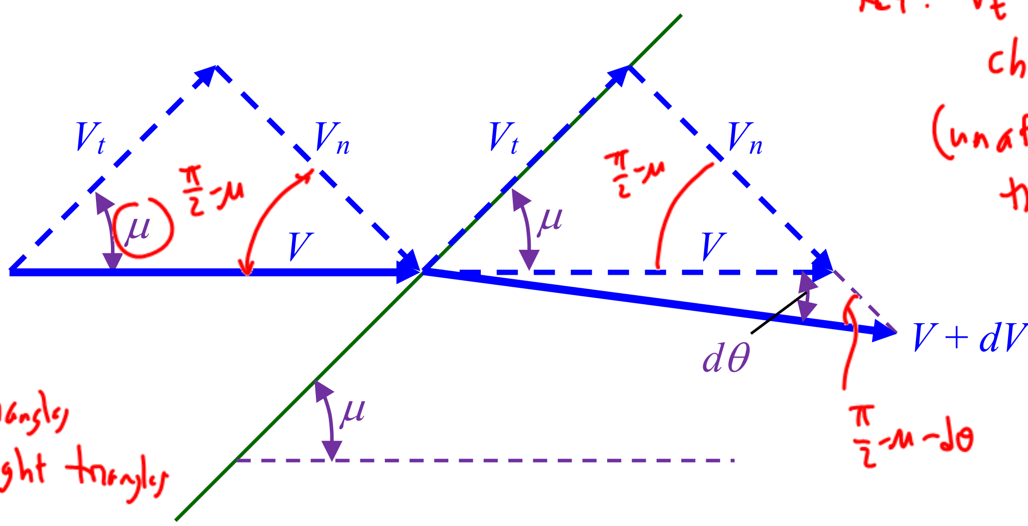
- ***Note***: If the corner is ***rounded***, we have the same expansion fan, but relative to a ***virtual origin*** under the wall.



But how do we calculate M_2?

- Consider one of the rays – one of the P-M waves with infinitesimal change in flow angle $d\theta$ and speed dV (with corresponding infinitesimal changes in other properties such as M, T, P, etc.):

KEY: V_t does not change (unaffected by the P-M wave)



All 3 triangles are right triangles

- Useful trig identities and series expansions:

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$ $\quad$ $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\dfrac{1}{1-x}=1+x+x^2+x \; +\dots$

$$\boxed{\frac{V_t}{V} = \sin\left(\frac{\pi}{2}-\mu\right)} \qquad \boxed{\frac{V_t}{V+dV} = \sin\left(\frac{\pi}{2}-\mu-d\theta\right)}$$

let $d\theta \to 0$ $\qquad$ equate V_t $\qquad$ use $\mu = \sin^{-1}\left(\frac{1}{M}\right)$

$$\boxed{d\theta = \sqrt{M^2-1}\,\frac{dV}{V}}$$

THIS HOLDS FOR ANY GAS (we have not made ideal gas approx. yet.)

INTEGRATE

$$\boxed{\int_{\theta_1}^{\theta_2} d\theta = \int_{M_1}^{M_2} \sqrt{M^2-1}\,\frac{dV}{V}}$$

we want in terms of Mach #

$$V = Ma$$

$$\ln V = \ln(M \cdot a) = \ln M + \ln a$$

differentiate

$$\boxed{\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}}$$

NOW – APPROX AS IDEAL GAS: $\rightarrow$ $\boxed{\frac{a_0}{a} = \sqrt{\frac{T_0}{T}} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{2}}}$

$$\frac{dV}{V} = \frac{dM}{M}\,\frac{1}{1+\frac{\gamma-1}{2}M^2}$$

$$\boxed{\theta_2 - \theta_1 = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M}}$$

We typ. set $\theta_1 = 0$

INTRODUCE THE PRANDTL - MEYER FUNCTION $\nu(M)$

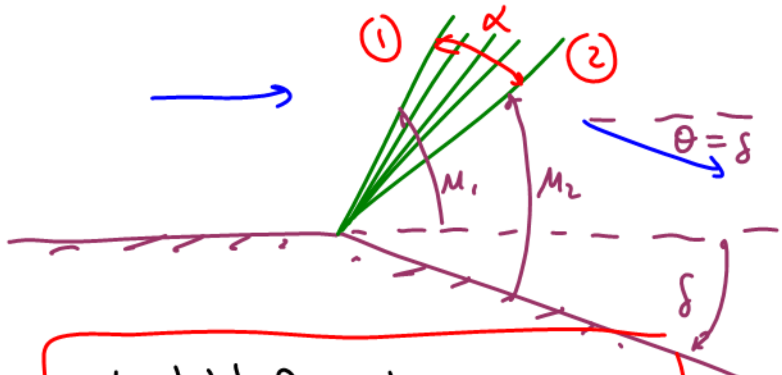
Not kinematic viscosity

$$\boxed{\nu(M) = \int \frac{\sqrt{M^2-1}}{1+\frac{\gamma-1}{2}M^2}\,\frac{dM}{M}} \Rightarrow \boxed{\theta_2 - \cancel{\theta_1} = \nu(M_2) - \nu(M_1)}$$

typ. set = 0

THE PRANDTL - MEYER EQ. $\boxed{\theta_2 = \nu(M_2) - \nu(M_1)}$

Apply:



α = total turning angle of the P-M exp. fan

$$\boxed{\alpha = \text{total fan angle} = \mu_1 - \mu_2 + \delta}$$

[$\delta = \theta_2$ here]

INTEGRATE (FOR IDEAL GAS)

Finally, after much algebra…

Final result for 2-D Prandtl-Meyer expansion fans:

$$\boxed{\theta_2 = \nu(M_2) - \nu(M_1)}$$

ν is an angle!

where we define:

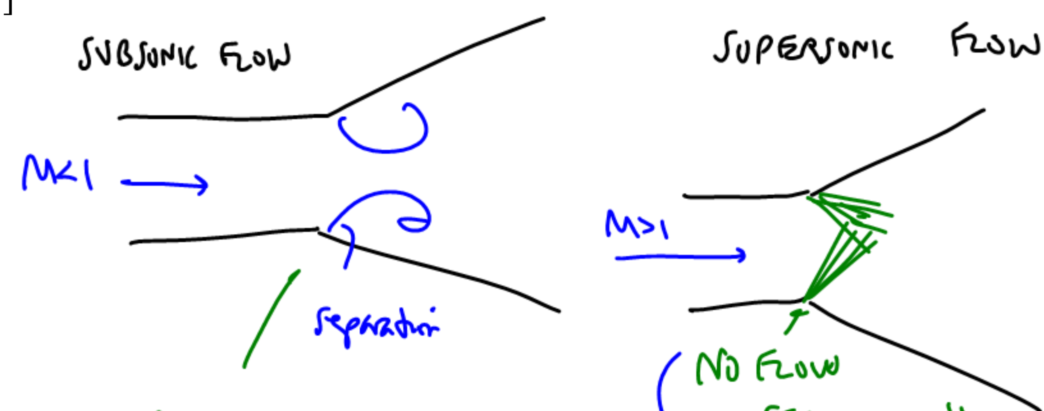
The Prandtl-Meyer Function (for an ideal gas):

P–M function: $$\boxed{\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\left[\sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)}\right] - \tan^{-1}\left[\sqrt{\left(M^2-1\right)}\right]}.$$

For air (γ =1.4): $$\boxed{\nu(M) = \sqrt{6}\tan^{-1}\left[\sqrt{\frac{1}{6}\left(M^2-1\right)}\right] - \tan^{-1}\left[\sqrt{\left(M^2-1\right)}\right]}.$$

[These three equations are on the equation sheet and are all sufficient to solve P-M expansion fan problems, along with isentropic relations to solve for other fluid properties like pressure, temperature, etc.]

Q



ADVERSE PRESSURE GRADIENT

V↓ P↑ M↓

FAVORABLE P.G.

V↑ P↓ M↑

E.g. If $M=2.0$ $\nu(2.0) = 26.38°$

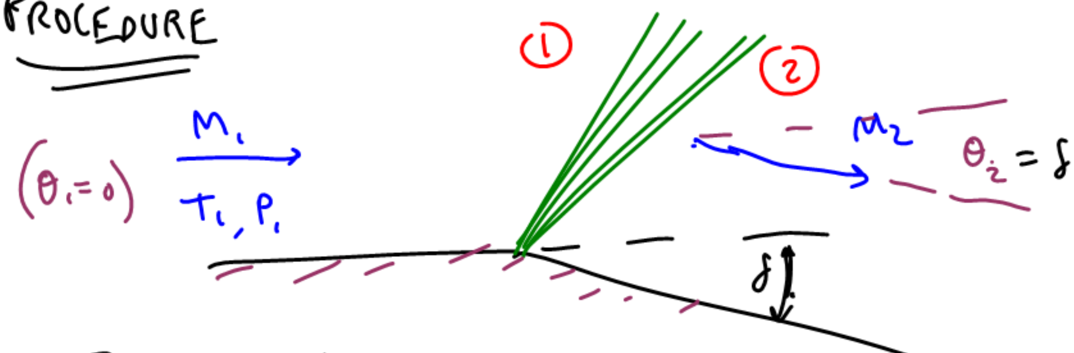
See CAC under isentropic flow

AGREES ☺

Comments about $\nu(M)$

- $\nu(M)$ is an angle
- $\nu(M)$ is not any physical angle in the problem
- When $M=1$, $\nu(M)=0$
- $\nu\uparrow$ as $M\uparrow$

Procedure



To calc. M_2, T_2, P_2

Procedure:

1) @ $M=M_1 \rightarrow$ calc $\nu(M_1)$ explicit
2) calc $\boxed{\nu(M_2) = \nu(M_1)+\theta_2}$

(Hard part) $\rightarrow$ 3) Calc M_2 = implicit fnc of $\nu(M_2)$

4) Use isentropic relations to get T_2, P_2, etc.

easy since isentropic!

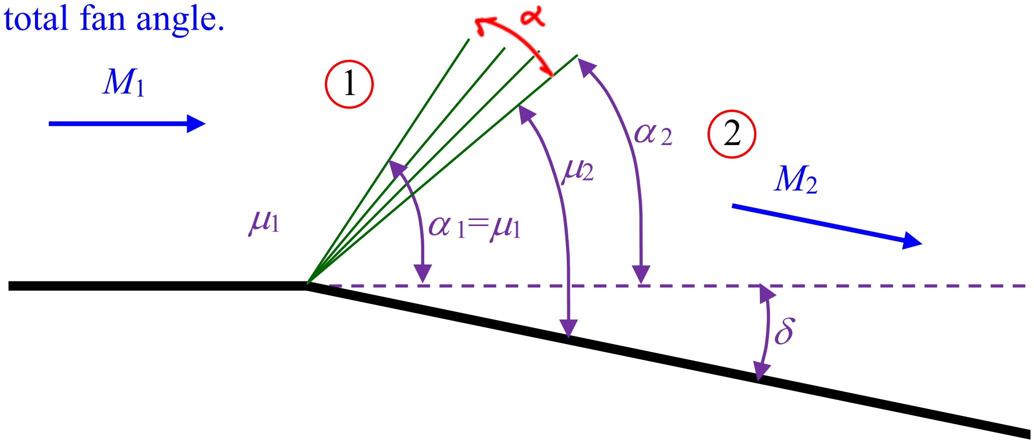
$T_0 = \text{const}$ $P_0 = \text{const}$

Example: P-M expansion fan

Given: Air flows supersonically and horizontally until it encounters a sudden expansion:

- $M_1 = 2.0$
- $P_1 = 230$ kPa and $T_1 = 300$ K
- Wall deflection angle $\delta = 10°$

To do: Calculate M_2, P_2, T_2, the angles of the leading and trailing waves of the expansion fan, and the total fan angle.



Solution:

Assumptions and Approximations:

1. The air is an ideal gas.
2. The process is adiabatic.
3. The process is isentropic (ignore boundary layers, friction, etc).

PROCEDURE:

1) @ $M_1 = 2.0$ _ $\nu(M_1) = 26.38°$

2) $\nu(M_2) = \nu(M_1) + \theta_2$

$= 26.38° + 10° = 36.38°$

3) Solve for M_2 @ $\nu(M_2) = 36.38°$

IMPLICIT ⟶ get $\boxed{M_2 = 2.385}$

4) Use isentropic relations

$$\boxed{P_2 = \frac{P_2}{P_1} P_1 = \frac{P_2}{P_0} \frac{P_0}{P_1} P_1} \rightarrow \boxed{P_2 = 126.0 \text{ kPa}}$$

Similarly for T_2 ⟶ $\boxed{T_2 = 252.6 \text{ K}}$

$$\mu_1 = \sin^{-1}\frac{1}{M_1} = 30^\circ$$

$$\mu_2 = \sin^{-1}\frac{1}{M_2} = 24.79^\circ$$

$$\alpha_{fin} = \text{total fin angle} = \mu_1 - \mu_2 + \delta$$

$$= 30 - 24.79 + 10 = 15.21^\circ$$

$$\boxed{\alpha = 15.2^\circ}$$