Equation Sheet for M E 521, *using notation of Kundu, Cohen, and Dowling Ed. 6*

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- **Tensor transformation rules for rotation**: $C_{ij} \equiv \cos\alpha_{ij}$, where α_{ij} = angle between the *old* (i) and *new* (j) axes. Then, for tensor A, $A'_m = C_{im}A_i$ $A'_{mn} = C_{im}C_{jn}A_{ij}$ $A'_{mnp} = C_{im}C_{jn}C_{kp}A_{ijk}$, etc.
- **Delta, epsilon, and epsilon-delta relation**: $\delta_{ij} = \vec{e}_i \cdot \vec{e}_j$ $\varepsilon_{ijk}\vec{e}_k = \vec{e}_i \times \vec{e}_j$ $\varepsilon_{ijk}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$
- **Dot product**: $\vec{a}\cdot\vec{b} = (a_i\vec{e}_i)\cdot(b_j\vec{e}_j) = a_i b_j(\vec{e}_i\cdot\vec{e}_j) = a_i b_j \delta_{ij}$ **Cross product**: $\vec{a}\times\vec{b} = \varepsilon_{ijk}a_i b_j \vec{e}_k$ or $(\vec{a}\times\vec{b})_k = \varepsilon_{ijk}a_i b_j$
- **Gauss (G), Stokes (S), and Leibniz (L) theorems**:

  G: $\int_V \frac{\partial F}{\partial x_i} dV = \oint_A F dA_i$ S: $\int_A (\vec{\nabla}\times\vec{u})\cdot d\vec{A} = \oint_C \vec{u}\cdot d\vec{s} = \Gamma$ L: $\frac{d}{dt}\int_{V(t)} F(\vec{x},t)dV = \int_{V(t)} \frac{\partial F(\vec{x},t)}{\partial t} dV + \oint_{A(t)} F(\vec{x},t)\vec{u}_A \cdot d\vec{A}$
- **Material derivative**: $\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u_j \frac{\partial F}{\partial x_j}$ (following a fluid particle) where F is some variable
- **Strain rate tensor**: $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ for any fluid. Principal strain rates (eigenvalues) found from $\det\left|S_{ij} - \lambda\delta_{ij}\right| = 0$
- **Reynolds transport theorem**: $\frac{D}{Dt}\int_{V(t)} F dV = \int_{CV} \frac{\partial F}{\partial t} dV + \oint_{CS} F u_j dA_j$ where F can be any quantity per unit volume
- **Conservation of mass**: $0 = \int_{CV} \frac{\partial \rho}{\partial t} dV + \oint_{CS} \rho u_j dA_j$ $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$
- **Linear momentum equation**:

  For *any* fluid: $\int_{CV} \frac{\partial}{\partial t}(\rho u_i) dV + \oint_{CS} \rho u_i u_j dA_j = \int_{CV} \rho g_i dV + \oint_{CS} T_{ij} dA_j$

  Differential form: $\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \rho g_i + \frac{\partial T_{ij}}{\partial x_j}$ or $\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial T_{ij}}{\partial x_j}$ (this equation is called **Cauchy's equation**)

  Constitutive equation (relation between stress and strain), with τ_{ij} defined as the *deviatoric stress tensor*: $T_{ij} = -p\delta_{ij} + \tau_{ij}$

  For *Newtonian* fluid: $T_{ij} = -p\delta_{ij} + 2\mu S_{ij} + \lambda S_{mm}\delta_{ij}$, and the famous **Navier-Stokes equation** results:

  For *compressible* flow: $\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \rho g_i + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \frac{\partial}{\partial x_i}\left(\lambda\frac{\partial u_j}{\partial x_j}\right)$

  For *incompressible* flow: $\tau_{ij} = 2\mu S_{ij}$ $T_{ij} = -p\delta_{ij} + 2\mu S_{ij}$ $\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$
- **Useful equations for incompressible flow in Cartesian coordinates (x, y, z), (u, v, w)**:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \vec{u}\cdot\vec{\nabla} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$S_{xx} = \frac{\partial u}{\partial x} = \frac{1}{2\mu}\tau_{xx} \qquad S_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2\mu}\tau_{xy} \qquad S_{yy} = \frac{\partial v}{\partial y} = \frac{1}{2\mu}\tau_{yy}$$

Vorticity: $\vec{\omega} = \vec{\nabla}\times\vec{u}$ or $\omega_k = \varepsilon_{ijk}\frac{\partial u_j}{\partial x_i}$, with components $\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$ $\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

- **Useful equations for incompressible flow in cylindrical coordinates (r,θ, z), (u_r, u_θ, u_z):**

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(u_\theta)+\frac{\partial}{\partial z}(u_z)=0 \qquad \nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2} \qquad \vec{u}\cdot\vec{\nabla}=u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}$$

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+g_r+\nu\left(\nabla^2u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)$$

$$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_ru_\theta}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+g_\theta+\nu\left(\nabla^2u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right)$$

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}+u_z\frac{\partial u_z}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+g_z+\nu\left(\nabla^2u_z\right)$$

$$S_{rr}=\frac{\partial u_r}{\partial r}=\frac{1}{2\mu}\tau_{rr} \qquad S_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}=\frac{1}{2\mu}\tau_{\theta\theta} \qquad S_{r\theta}=\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{2r}\frac{\partial u_r}{\partial\theta}=\frac{1}{2\mu}\tau_{r\theta}$$

Vorticity: $\vec{\omega}=\vec{\nabla}\times\vec{u}$, with components $\omega_r=\frac{1}{r}\frac{\partial u_z}{\partial\theta}-\frac{\partial u_\theta}{\partial z}$ $\omega_\theta=\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}$ $\omega_z=\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)-\frac{1}{r}\frac{\partial u_r}{\partial\theta}$

- **Mechanical energy equation**: $\frac{\partial}{\partial t}\int_{CV}\left(\tfrac{1}{2}\rho u_iu_i\right)dV+\oint_{CS}\left(\tfrac{1}{2}\rho u_iu_iu_j\right)dA_j=\int_{CV}\rho g_iu_idV+\oint_{CS}T_{ij}u_idA_j+\int_{CV}p\frac{\partial u_i}{\partial x_i}dV-\int_{CV}\phi dV$

Differential form: $\frac{\partial\left(\frac{1}{2}\rho u_iu_i\right)}{\partial t}+\frac{\partial}{\partial x_j}\left(u_j\tfrac{1}{2}\rho u_iu_i\right)=\rho u_ig_i+\frac{\partial}{\partial x_j}\left(T_{ij}u_i\right)+p\frac{\partial u_j}{\partial x_j}-\phi$ where the rate of viscous dissipation of kinetic energy per unit volume is $\phi\equiv\tau_{ij}\,\partial u_i/\partial x_j$ where deviatoric stress tensor = $\tau_{ij}=T_{ij}+p\delta_{ij}$

[*Note*: Kundu's textbook uses $\varepsilon\equiv\tau_{ij}S_{ij}/\rho$ as the rate of kinetic energy dissipation per unit *mass*.]

- **First law (heat equation)**: Note that in the text, *e* (rather than the usual *u*) is the internal energy per unit mass.

$$\int_{CV}\frac{\partial}{\partial t}\left[\rho\left(e+\tfrac{1}{2}u_iu_i\right)\right]dV+\oint_{CS}\rho\left(e+\tfrac{1}{2}u_iu_i\right)u_jdA_j=\int_{CV}\rho g_iu_idV+\oint_{CS}T_{ij}u_idA_j-\oint_{CS}q_idA_i$$

Differential form: $\rho\frac{D}{Dt}\left(e+\tfrac{1}{2}u_iu_i\right)=\rho u_ig_i+\frac{\partial}{\partial x_j}\left(T_{ij}u_i\right)-\frac{\partial q_i}{\partial x_i}$ $\rho\frac{De}{Dt}=-\frac{\partial q_i}{\partial x_i}-p\frac{\partial u_i}{\partial x_i}+\phi$

If *incompressible*: $\rho C_p\frac{DT}{Dt}=k\frac{\partial^2T}{\partial x_i\partial x_i}+2\mu S_{ij}S_{ij}$ If *ideal gas*: $\rho C_p\frac{DT}{Dt}=\frac{Dp}{Dt}+\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)+\phi$

If *ideal gas at very low Mach number*: $\rho C_p\frac{DT}{Dt}\approx k\frac{\partial^2T}{\partial x_i\partial x_i}$

Exam 1 material ends here.

- **The *T-ds* equations of thermodynamics**: $Tds=de+pd\upsilon$ $Tds=dh-\upsilon dp$ where T = temperature, p = pressure, e = specific internal energy, h = specific enthalpy, s = specific entropy, and $\upsilon=1/\rho$ = specific volume
- **Bernoulli equations**: [These are the most common ones; there are many more forms not listed here.]

For *incompressible, steady, irrotational flow*: $\frac{p}{\rho}+\tfrac{1}{2}V^2+gz=\text{constant}$ where $V^2=u_ju_j=|\vec{u}|^2$

For *incompressible, steady, inviscid flow*: $\frac{p}{\rho}+\tfrac{1}{2}V^2+gz=\text{constant along a streamline}$

For *steady, compressible, inviscid, irrotational, isentropic flow*: $h+\tfrac{1}{2}V^2+gz=\text{constant}$ where $h\equiv e+\frac{p}{\rho}$

- **Boussinesq Approximation**: $\rho=\rho_o\left[1-\alpha\left(T-T_o\right)\right]$ where $\alpha=-\frac{1}{\rho}\left.\frac{\partial\rho}{\partial T}\right)_p$ = coefficient of thermal expansion and the z-momentum equation is $\frac{Dw}{Dt}=\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{1}{\rho_o}\frac{\partial p}{\partial z}-\left[1-\alpha\left(T-T_o\right)\right]g+\nu\left(\frac{\partial^2w}{\partial x^2}+\frac{\partial^2w}{\partial y^2}+\frac{\partial^2w}{\partial z^2}\right)$

- **Interaction of vortices**: Velocity induced by one vortex on its neighbors: $u_\theta = \dfrac{\Gamma}{2\pi r}$ where $\Gamma \equiv \text{ circulation } = \oint_C \vec{u} \cdot \vec{ds}$

- **Vorticity equation for incompressible Newtonian flow**: $\vec{\omega} = \vec{\nabla} \times \vec{u}$ $\dfrac{D\omega_k}{Dt} = \omega_j \dfrac{\partial u_k}{\partial x_j} + \nu \dfrac{\partial^2 \omega_k}{\partial x_j \partial x_j}$

- **Two-dimensional potential flow**: $\vec{u} = \vec{\nabla}\phi$ $u = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$ $v = \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x}$ $u_r = \dfrac{\partial \phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}$ $u_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} = -\dfrac{\partial \psi}{\partial r}$

  $z \equiv x + iy = re^{i\theta}$ $z^* \equiv x - iy = re^{-i\theta}$ $i \equiv \sqrt{-1}$ $r = |z| = \sqrt{x^2 + y^2}$ $|z| = \sqrt{zz^*}$ $e^{i\theta} = \cos\theta + i\sin\theta$ $e^{-i\theta} = \cos\theta - i\sin\theta$

  Complex potential: $w(z) = \phi + i\psi$ Complex velocity: $\dfrac{dw}{dz} = u(x,y) - iv(x,y) = \left(u_r(r,\theta) - iu_\theta(r,\theta)\right)e^{-i\theta}$ Uniform stream at angle of attack: $w = Uze^{-i\alpha}$ Line source: $w = \dfrac{m}{2\pi}\ln z$ Line vortex: $w = -i\dfrac{\Gamma}{2\pi}\ln z$ Doublet: $w = \dfrac{\mu}{z}$ where $\mu = \dfrac{m\varepsilon}{\pi}$ Power function (stagnation point, corner flows, etc.): $w = Az^n$ Lift per unit span due to a bound vortex on a closed 2-D body: $\rho U \Gamma_a$

- **Zhukhovsky transformation**: $z = \zeta + \dfrac{b^2}{\zeta}$ transforms circle of radius b in ζ-plane into line of length $4b$ in z-plane

- **Axisymmetric potential flow**: $\vec{u} = \vec{\nabla}\phi$ $u_r = \dfrac{\partial \phi}{\partial r} = \dfrac{1}{r^2 \sin\theta}\dfrac{\partial \psi}{\partial \theta}$ $u_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} = \dfrac{-1}{r\sin\theta}\dfrac{\partial \psi}{\partial r}$ Uniform stream: $\psi = \frac{1}{2}Ur^2 \sin^2\theta$

  Point source: $\psi = -\dfrac{Q}{4\pi}\cos\theta$ Doublet: $\psi = -\dfrac{m}{r}\sin^2\theta$

- **Induced drag** (on a finite wing): $D_i = \int_A \frac{1}{2}\rho V^2 dA$ where $V^2 = v^2 + w^2$ in plane A somewhere far downstream of the wing

- **One-dimensional diffusion equation**: $\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2}$ vorticity diffusion from a wall goes like $\delta \sim \sqrt{\nu t}$

- **Stokes first problem** (impulsively started plate): $\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2}$ $\eta = \dfrac{y}{\delta(t)}$ $F(\eta) = \dfrac{u}{U}$ yields $\dfrac{\delta}{\nu}\dfrac{d\delta}{dt} = \dfrac{-F''}{\eta F'} = c$ with solution $\delta = \sqrt{2c\nu t}$ and $F(\eta) = 1 - \text{erf}\left(\sqrt{\dfrac{c}{2}}\eta\right)$ or $\dfrac{u}{U} = 1 - \text{erf}\left(\dfrac{y}{2\sqrt{\nu t}}\right)$ and $\omega_z = \dfrac{U}{\sqrt{\pi \nu t}}\exp\left(\dfrac{-y^2}{4\nu t}\right)$

- **Viscous decay (diffusion) of a line vortex**: $\dfrac{\partial u_\theta}{\partial t} = \nu \dfrac{\partial}{\partial r}\left[\dfrac{1}{r}\dfrac{\partial (ru_\theta)}{\partial r}\right]$ $u_\theta = \dfrac{\Gamma}{2\pi r}$ at $t = 0$ $u_\theta(r,t) = \dfrac{\Gamma}{2\pi r}\left[1 - e^{\frac{-r^2}{4\nu t}}\right]$

Exam 2 material ends here.

- **2-D stagnation point flow similarity solution**: $\psi = BxF(y)$ $u = BxF'(y)$ $v = -BF(y)$ $f(\eta) = F(y)\sqrt{B/\nu}$ $\eta = y\sqrt{B/\nu}$ yields $f''' + ff'' + 1 - (f')^2 = 0$

- **Stokes flow (creeping flow)**: $\vec{\nabla} \cdot \vec{u} = 0$ $\vec{\nabla}p = \mu \nabla^2 \vec{u}$ (neglecting the gravity term) $\text{Drag} \sim \mu UL$ for 3-D body of length L. For a sphere, $D = 6\pi\mu Ua = 3\pi\mu UD_p$

- **2-D (r,θ) incompressible vorticity equation in z-direction**: $\dfrac{\partial \omega_z}{\partial t} + u_r \dfrac{\partial \omega_z}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial \omega_z}{\partial \theta} = \nu\left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \omega_z}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \omega_z}{\partial \theta^2}\right]$

- **Two-D incompressible laminar boundary layers**: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ $\dfrac{\partial p}{\partial y} \approx 0$ $\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \dfrac{\partial U}{\partial t} + U\dfrac{\partial U}{\partial x} + \nu \dfrac{\partial^2 u}{\partial y^2}$

- **Blasius flat plate boundary layer (U = constant, $dp/dx = 0$)**: $u = Uf'(\eta)$ $\eta = y\sqrt{\frac{U}{\nu x}}$ $f''' + cff'' = 0$, where we pick $c = ½$.

  $v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\left(\eta f' - f\right) = 0.860\sqrt{\frac{\nu U}{x}}$ as $\eta \to \infty$ $\frac{\delta_{0.99}}{x} = \frac{4.91}{\sqrt{\text{Re}_x}}$ $\frac{\delta^*}{x} = \frac{\int_0^\infty \left(1 - \frac{u}{U}\right)dy}{x} = \frac{1.72}{\sqrt{\text{Re}_x}}$ $\frac{\theta}{x} = \frac{\int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)dy}{x} = \frac{0.664}{\sqrt{\text{Re}_x}}$

  where $\text{Re}_x = \frac{\rho U x}{\mu} = \frac{Ux}{\nu}$. Local skin friction coefficient at some x location: $C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{\text{Re}_x}}$. Drag coefficient per unit depth of the plate on one side of the plate only at some x location: $C_D = \frac{D}{\frac{1}{2}\rho U^2 x} = \frac{1.33}{\sqrt{\text{Re}_x}}$.

- **Falkner-Skan wedge flow boundary layer similarity solution**: $U(x) = Bx^m$ $u = Uf'(\eta)$ $\eta = \frac{y}{\delta_c(x)}$ yields

  $f''' + ff'' + \beta[1 - (f')^2] = 0$ where $\beta \equiv \frac{\delta_c^{\,2}}{\nu}\frac{dU}{dx}$ and $b = \frac{1-m}{2}$, $C = \sqrt{\frac{\nu(2-\beta)}{B}}$, $\beta = \frac{2m}{1+m}$ or $m = \frac{\beta}{2-\beta}$