Equation Sheet for ME 522

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Equation Sheet for homework, Candy Friday questions, midterm exams, and the final exam.

- **Two-dimensional boundary layer (BL) equations:** $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$, $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial p}{\partial y}=0$.
- **Mangler's axisymmetric BL equations:** $\frac{\partial(r_0 u)}{\partial x}+r_0\frac{\partial v}{\partial y}=0$, $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial p}{\partial y}=0$.
- **General axisymmetric BL equations:** $\frac{\partial u_x}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(ru_r)=0$, $u_x\frac{\partial u_x}{\partial x}+u_r\frac{\partial u_x}{\partial r}=U\frac{dU}{dx}+\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right)$, $\frac{\partial p}{\partial r}=0$.
- **The Mangler transformation:** $x'=\frac{1}{L^2}\int_0^x r_0^{\,2}dx$, $y'=\frac{r_0 y}{L}$, $u'=u$, $U'(x')=U(x)$, $v'=\frac{L}{r_0}\left(v+\frac{dr_0}{dx}\frac{yu}{r_0}\right)$,

  which yields an *equivalent 2-D flow* with BL equations: $\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0$, $u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=U'\frac{dU'}{dx'}+\nu\frac{\partial^2 u'}{\partial y'^2}$.
- **Three-dimensional boundary layers:**

  Scale factors or stretching factors: For $\vec{R}\equiv(X,Y,Z)=\vec{r}(x,z)+y\vec{n}(x,z)$, $h_x\equiv\left|\frac{d\vec{R}}{dx}\right|=\sqrt{\left(\frac{\partial X}{\partial x}\right)^2+\left(\frac{\partial Y}{\partial x}\right)^2+\left(\frac{\partial Z}{\partial x}\right)^2}$,

  $h_y\equiv\left|\frac{d\vec{R}}{dy}\right|=\sqrt{\left(\frac{\partial X}{\partial y}\right)^2+\left(\frac{\partial Y}{\partial y}\right)^2+\left(\frac{\partial Z}{\partial y}\right)^2}$, $h_z\equiv\left|\frac{d\vec{R}}{dz}\right|=\sqrt{\left(\frac{\partial X}{\partial z}\right)^2+\left(\frac{\partial Y}{\partial z}\right)^2+\left(\frac{\partial Z}{\partial z}\right)^2}$. ***The 3-D BL equations***:

  $\frac{1}{h_x h_z}\left[\frac{\partial}{\partial x}(h_z u)+\frac{\partial}{\partial z}(h_x w)\right]+\frac{\partial v}{\partial y}=0$, $\frac{u}{h_x}\frac{\partial u}{\partial x}+\frac{w}{h_z}\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial y}+\frac{uw}{h_x h_z}\frac{\partial h_x}{\partial z}-\frac{w^2}{h_x h_z}\frac{\partial h_z}{\partial x}=-\frac{1}{\rho h_x}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial p}{\partial y}=0$,

  $\frac{u}{h_x}\frac{\partial w}{\partial x}+\frac{w}{h_z}\frac{\partial w}{\partial z}+v\frac{\partial w}{\partial y}-\frac{u^2}{h_x h_z}\frac{\partial h_x}{\partial z}+\frac{uw}{h_x h_z}\frac{\partial h_z}{\partial x}=-\frac{1}{\rho h_z}\frac{\partial p}{\partial z}+\nu\frac{\partial^2 w}{\partial y^2}$.
- **Some complex variable definitions:** For real variable x, $e^{ix}=\cos x+i\sin x$, $e^{-ix}=\cos x-i\sin x$, $\cosh(x)=\frac{e^x+e^{-x}}{2}$, $\sinh(x)=\frac{e^x-e^{-x}}{2}$. For complex variable $z=x+iy$, the sine and cosine of z are $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$ and $\cos z=\frac{e^{iz}+e^{-iz}}{2}$, and the hyperbolic functions are defined as $\cosh(z)=\frac{e^z+e^{-z}}{2}$ and $\sinh(z)=\frac{e^z-e^{-z}}{2}$.
- **Linear stability analysis:** *Notation*: $\tilde{q}=Q+q$, where $\tilde{q}$ = total flow variable, Q = basic state, and q = disturbance. [For temperature and density, use $\tilde{T}=\bar{T}+T'$ and $\tilde{\rho}=\bar{\rho}+\rho$.] A variable with a hat like $\hat{q}$ is an *amplitude*.

  Procedure: Step 0: Eq. of motion $\mathcal{D}(\tilde{q})=0$. Step 1: Basic state $\mathcal{D}(Q)=0$. Step 2: Add disturbance $\mathcal{D}(Q+q)=0$. Step 3: Subtract basic state $\mathcal{D}(Q+q)-\mathcal{D}(Q)=0$. Step 4: Linearize. Step 5: Solve linearized disturbance eq. for q. Step 6: Examine stability.
- **Method of normal modes:** $q(x,y,z,t)=\hat{q}(z)e^{ikx+ily+\sigma t}$ for wave disturbances in the x and/or y directions, where q is a disturbance quantity – typically u, v, w, and p, or sometimes disturbance stream function ψ, and $\hat{q}$ is an amplitude.

Exam 1 material ends here.

- **Stability of locally parallel flows:** *Basic state*: $U=U(y)$, and $V=W=0$ [flow in x direction only].

  Normal modes: $\psi(x,y,t)=\phi(y)e^{ik(x-ct)}$, where ψ is the *disturbance stream function*, $\phi(y)$ is the amplitude of the disturbance, k is the wave number, and c is the wave speed. In general, k and c may be complex.

  This yields the ***Orr-Sommerfeld equation***: $(U-c)\left(\phi_{yy}-k^2\phi\right)-U_{yy}\phi=\frac{1}{ik\,\mathrm{Re}}\left[\phi_{yyyy}-2k^2\phi_{yy}+k^4\phi\right]$.

Turbulence, the Final Frontier!

- **Boussinesq total flow equations:** $\frac{\partial \tilde{u}_i}{\partial x_i}=0$, $\left(\frac{\partial \tilde{u}_i}{\partial t}+\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial x_j}\right)=-\frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial x_i}-g\delta_{i3}\left[1-\alpha\left(\tilde{T}-T_0\right)\right]+\nu\frac{\partial^2\tilde{u}_i}{\partial x_j\partial x_j}$,
  $\frac{\partial \tilde{T}}{\partial t}+\tilde{u}_j\frac{\partial \tilde{T}}{\partial x_j}=\kappa\frac{\partial^2\tilde{T}}{\partial x_j\partial x_j}$. [*Note*: These eqs. include buoyancy, but we usually ignore gravitational effects.]
- **Reynolds decomposition:** $\tilde{q}=Q+q$, $\tilde{u}_i=U_i+u_i$, $\tilde{p}=P+p$, etc. **Reynolds stress tensor:** $\mathrm{RS}\equiv-\rho\overline{u_iu_j}$.
- **Incompressible mean flow:** Continuity: $\frac{\partial U_i}{\partial x_i}=0$, Momentum: $\rho\left(\frac{\partial U_i}{\partial t}+U_j\frac{\partial U_i}{\partial x_j}\right)=-\frac{\partial P}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial U_i}{\partial x_j}-\rho\overline{u_iu_j}\right)$,
  Mean Kinetic Energy Equation: $\frac{D}{Dt}\left(\tfrac{1}{2}U_i^2\right)=\frac{\partial}{\partial x_j}\left[-\frac{PU_j}{\rho_0}+2\nu U_iS_{ij}-\overline{u_iu_j}U_i\right]+\overline{u_iu_j}\frac{\partial U_i}{\partial x_j}-\frac{g}{\rho_0}\overline{\rho}U_3-2\nu S_{ij}S_{ij}$.
- **Mean strain rate tensor:** $S_{ij}\equiv\tfrac{1}{2}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)$. **Fluctuating strain rate tensor:** $S_{ij}'\equiv\tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$.
- **Kinematic Reynolds stress transport equation:**
  $$\frac{D}{Dt}\left(\overline{u_iu_j}\right)=-\left(\overline{u_iu_k}\frac{\partial U_j}{\partial x_k}+\overline{u_ju_k}\frac{\partial U_i}{\partial x_k}\right)+\overline{\frac{p}{\rho}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}-\frac{\partial}{\partial x_k}\left[\overline{u_iu_ju_k}-\nu\frac{\partial\left(\overline{u_iu_j}\right)}{\partial x_k}+\overline{\frac{p}{\rho}\left(u_i\delta_{jk}+u_j\delta_{ik}\right)}\right]-2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}.$$
- **Turbulent kinetic energy per unit mass (t.k.e.):** $q^2=k=K\equiv\tfrac{1}{2}\overline{u_iu_i}$. [I will use K for these equations.]
- **The tke equation:** $\frac{D}{Dt}(K)=\frac{\partial}{\partial x_j}\left[-\frac{1}{\rho_0}\overline{pu_j}-\tfrac{1}{2}\overline{u_i^2u_j}+2\nu\overline{u_iS_{ij}'}\right]-\overline{u_iu_j}\frac{\partial U_i}{\partial x_j}+g\alpha\overline{wT'}-2\nu\overline{S_{ij}'S_{ij}'}$.
- **Turbulent Re:** $R_\ell\sim\frac{u'\ell}{\nu}$. **Kolmogorov microscales:** $\upsilon\sim(\nu\varepsilon)^{1/4}$, $\eta\sim\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$, $\tau\sim\left(\frac{\nu}{\varepsilon}\right)^{1/2}$, where $\varepsilon\sim\frac{u'^3}{\ell}$.

Exam 2 material ends here.

- **Eddy viscosity model for a 2-D Boundary Layer type flow:** $-\rho\overline{uv}=\mu_e\frac{dU}{dy}$ or $-\overline{uv}=\nu_e\frac{dU}{dy}$.
- **Standard high Reynolds number form of the *K*-ε turbulence model:**
  $\frac{DK}{Dt}\equiv\frac{\partial K}{\partial t}+U_j\frac{\partial K}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\frac{\nu_e}{\sigma_k}\frac{\partial K}{\partial x_j}\right)+\nu_e\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}-\varepsilon$ $\nu_e=\frac{\mu_e}{\rho}=C_\mu\frac{K^2}{\varepsilon}$
  $\frac{D\varepsilon}{Dt}\equiv\frac{\partial\varepsilon}{\partial t}+U_j\frac{\partial\varepsilon}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\frac{\nu_e}{\sigma_\varepsilon}\frac{\partial\varepsilon}{\partial x_j}\right)+\frac{C_{\varepsilon_1}\nu_e\varepsilon}{K}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}-C_{\varepsilon_2}\frac{\varepsilon^2}{K}$. The constants are $\sigma_k=1.0$, $\sigma_\varepsilon=1.3$,
  $C_\mu=0.09$, $C_{\varepsilon_1}=1.44$, and $C_{\varepsilon_2}=1.92$. The ***Boussinesq eddy viscosity model*** is $-\rho\overline{u_iu_j}\approx-\tfrac{2}{3}\rho K\delta_{ij}+2\mu_eS_{ij}$.
- **Approximate relations for a turbulent flat plate boundary layer:** $C_f=\frac{2\tau_w}{\rho U^2}=0.027\,\mathrm{Re}_x^{-1/7}$,
  $C_D=\frac{2D}{\rho U^2bx}=0.031\,\mathrm{Re}_x^{-1/7}$, $\frac{\delta}{x}=0.16\,\mathrm{Re}_x^{-1/7}$.
- **Inner variables (wall variables) for BL, channel, and pipe flows:** $\tau_w\equiv\mu\left.\frac{dU}{dy}\right)_w$, $u^*\equiv\sqrt{\frac{\tau_w}{\rho}}$, $u^+=\frac{U}{u^*}$, $y^+\equiv\frac{y_wu^*}{\nu}$
  where y_w is the distance from the wall ($y_w=y$ for a BL; $y_w=y+b$ for a channel).
- **Viscous sublayer:** $u^+=y^+$. **Log-law layer:** $u^+=\frac{1}{\kappa}\ln y^++B$ where κ = 0.40 to 0.41, B (also called a) = 5.0 to 5.5.