Due: In class, Friday **February 1, 2019**	**Name**	**PSU ID (abc1234)**

ME 522

Spring Semester, 2019

Homework Set # 2

Professor J. M. Cimbala

For instructor or TA use only:		
Problem	**Score**	**Points**
1		20
2		20
3		20
4		20
5		20
Total:		**100**

1. (20 pts) In a class example, we examined entrainment of a 2-D laminar jet spilling polluted water into a large lake. Consider instead an ***axisymmetric* laminar jet**, with initial diameter d and uniform jet velocity U_0, exiting into the lake.

(a) Derive an expression for the downstream distance x required such that the jet contains only 1% original jet fluid and 99% ambient fluid.

(b) Plug in some numbers (use the same numbers as in the 2-D case), namely U_0 = 10. cm/s, d = 5.0 cm, and set the kinematic viscosity of both the effluent and the water to 0.0100 cm^2/s. Estimate x in meters and miles.

(c) Compare with the 2-D case. Is the axisymmetric result more realistic? *Is* it realistic? Why or why not?

(d) Later on in the course we will define a "turbulent" or "eddy viscosity" μ_t. This is an approximation that lets us treat the effect of turbulent eddies like an increased viscosity to take into account the much more rapid diffusion of mass, momentum, and heat due to turbulent eddies compared to laminar diffusion due simply to viscosity. [We "pretend" we have a laminar flow, but we use μ_t instead of μ as the viscosity.] Recall the class demonstration with a hair dryer. We found that the 99% diffusion distance of an actual round jet is only about two meters for our hair dryer, which is around 50 jet diameters. For the values given in this problem and assuming 50 diameters as the 99% diffusion distance, estimate the eddy viscosity ratio μ_t/μ (or equivalently ν_t/ν) that would yield the correct experimental diffusion distance for a turbulent jet. In other words, use the laminar solution, but with ν_t instead of ν as the kinematic viscosity of the fluid. Is it a good approximation to neglect ν compared to ν_t in a turbulent free shear flow calculation?

2. (20 pts) It is my understanding that students in ME 521 Fall 2018 did not do Runge-Kutta (R-K) solutions for any of the similarity problems. We will need this technique for ME 522. So… to get you some practice, let's consider the 2-D jet similarity flow that we discussed in class. Schlichting found an analytical solution, but we will solve it using R-K. For your convenience, I have posted three resources on the course website www.mne.psu.edu/cimbala/me522/ : (1) my Matlab solution for the Blasius problem, posted on the home page, which can be used as a guide or template if you wish, (2) My Excel file, also posted, and (3) a description of the R-K technique, posted in the tab called References.

(a) Solve this similarity problem using R-K. You may use any software you like; it is easiest to start with my Blasius case.

(b) Compare with the analytical solution given in class. Specifically, plot u/u_{max} as a function of η. For consistency, make η the *vertical* axis since it is "up." Only the upper half of the jet needs to be plotted because of symmetry. If you do the R-K solution correctly, your results should agree almost exactly with the analytical solution.

3. (20 pts) We will need to use complex variables late on in this course when we discuss stability analysis. So, here are a few "warm-up" exercises with complex variables. First some definitions:

For real variable x, $\boxed{e^{ix} = \cos x + i\sin x}$, $\boxed{e^{-ix} = \cos x - i\sin x}$, $\boxed{\cosh(x) = \frac{e^{x}+e^{-x}}{2}}$, and $\boxed{\sinh(x) = \frac{e^{x}-e^{-x}}{2}}$.

For complex variable z, where $z = x + iy$, the sine and cosine of z are $\boxed{\sin z = \frac{e^{iz}-e^{-iz}}{2i}}$ and $\boxed{\cos z = \frac{e^{iz}+e^{-iz}}{2}}$, and the hyperbolic sine and cosine functions are defined as $\boxed{\cosh(z) = \frac{e^{z}+e^{-z}}{2}}$ and $\boxed{\sinh(z) = \frac{e^{z}-e^{-z}}{2}}$.

(a) Show that $\boxed{\sin x = \frac{e^{ix}-e^{-ix}}{2i}}$ and $\boxed{\cos x = \frac{e^{ix}+e^{-ix}}{2}}$

(b) If k is a complex wave number, $k = k_r + ik_i$, show that $\boxed{e^{ikx} = e^{-k_i x}[\cos(k_r x) + i\sin(k_r x)]}$

(c) Show that $\boxed{\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)}$

(d) Find a similar expression for cos(z).

(e) Find an equation for sinh(iz) as a function of sin(z).

Note: There is another page. →

4. (20 pts) Consider general incompressible, steady, axisymmetric boundary layer flow without gravity. Do *not* use Mangler's transformation (there is no body). Start with the Navier-Stokes equations shown below in cylindrical coordinates (r,θ,x), (u_r,u_θ,u_x), where we leave out the θ-component of momentum since we are analyzing axisymmetric flow here, so $\partial(\text{anything})/\partial\theta = 0$ and u_θ is zero everywhere (no swirl). Note that we use x instead of z as the axis of symmetry.

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(u_\theta)+\frac{\partial}{\partial x}(u_x)=0} \quad \boxed{u_r\frac{\partial u_x}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_x}{\partial\theta}+u_x\frac{\partial u_x}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right)+\frac{\nu}{r^2}\frac{\partial^2 u_x}{\partial\theta^2}+\nu\frac{\partial^2 u_x}{\partial x^2}}$$

$$\boxed{u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}+u_x\frac{\partial u_r}{\partial x}-\frac{u_\theta^{\,2}}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)+\frac{\nu}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\nu\frac{\partial^2 u_r}{\partial x^2}-\nu\frac{u_r}{r^2}-\frac{2\nu}{r^2}\frac{\partial u_\theta}{\partial\theta}}$$

Let U be a characteristic streamwise velocity, and let δ_c be a characteristic boundary layer thickness. We shall reduce the general equations into axisymmetric boundary layer equations using order of magnitude (o.o.m.) estimates of each term, as we did in ME 521 for the 2-D boundary layer case. Namely, $\frac{\partial}{\partial x}\sim\frac{1}{x}$, $\frac{\partial}{\partial r}\sim\frac{1}{\delta_c}$, $u_x\sim U$, and changes in pressure $\sim\rho U^2$.

(**a**) Use continuity to find an o.o.m. estimate for u_r, and write the boundary layer form of the continuity equation.

(**b**) Perform a careful o.o.m. analysis of each term in the x-momentum equation, showing all your work. Write out the boundary layer form of the x-momentum equation. Also, in the process, find an o.o.m. estimate for δ_c as a function of Re_x, where $\boxed{\mathrm{Re}_x = Ux/\nu}$.

(**c**) Perform a careful o.o.m. analysis of each term in the r-momentum equation, showing all your work. Write out the boundary layer form of the r-momentum equation.

(**d**) Finally, summarize all three equations and make sure they match the ones given in class (and on the equation sheet).

5. (20 pts) In class we discussed the 3-D boundary layer equations, and defined ***scale factors*** (also called ***stretching factors***) h_x, h_y, and h_z. As a class example, we showed that for a cone, $h_x = 1$, $h_y = 1$, and $h_z = r$ when we let z be the same as θ. For a very *thin* boundary layer, it is reasonable to replace r by $r_0(x)$ everywhere inside the boundary layer.

(**a**) Plug these scale factors and the approximation for r into the 3-D boundary layer equations (see class notes for the full equations), and thereby derive the equations for rotationally symmetric, very thin boundary layer flow over a cone. You may assume that $\frac{\partial}{\partial\theta}=\frac{\partial}{\partial z}=0$ here since it is rotationally symmetric, but ***allow there to be swirl***, i.e., $w\neq 0$.

(**b**) Now let $w = 0$ (axisymmetric flow, no swirl), and show that the above equations reduce to ***Mangler's axisymmetric boundary layer equations***. It turns out that (you do not need to prove this), although derived here for a *cone*, these equations are valid for *any arbitrary shaped axisymmetric body with a thin boundary layer* (negligible transverse curvature effects)!