Due: In class, Friday **March 1, 2019**	**Name**	**PSU ID (abc1234)**

ME 522
Spring Semester, 2019
Homework Set # 5

Professor J. M. Cimbala

For instructor or TA use only:		
Problem	**Score**	**Points**
1		40
2		20
3		25
4		15
Total:		**100**

1. (40 pts) Reconsider the thermal instability (Bénard) problem we discussed in class. We stated without proof that the *even* mode was more likely to occur than the *odd* mode. Rather than trying to prove this mathematically, we will instead calculate the critical Rayleigh number for the odd mode, and compare it to that of the even mode. Of course, the mode with the smallest critical value of Rayleigh number "wins", and that mode is the one we would expect to see, which is the even mode. Start with the solution of the differential equation for modified amplitude $W(z^*)$ that we generated in class,

$$\boxed{W\left(z^*\right) = C_1 e^{iq_0 z^*} + C_2 e^{-iq_0 z^*} + C_3 e^{qz^*} + C_4 e^{-qz^*} + C_5 e^{q^* z^*} + C_6 e^{-q^* z^*}} \tag{1}$$

We considered only the even mode in class, and we re-grouped the six terms in the above equation into a simplified equation with only three constants, A, B, and C, in terms of cosines and hyperbolic cosines, namely,

$$\boxed{W\left(z^*\right) = A\cos\left(q_0 z^*\right) + B\cosh\left(qz^*\right) + C\cosh\left(q^* z^*\right)} \tag{2}$$

(a) Using the *same notation* as we did in the lecture notes for the even mode, consider the *odd mode*. Re-group the terms in the above equation for the odd mode and generate a simplified equation with only three constants, (for consistency, again call them A, B, and C), in terms of sines and hyperbolic sines, showing your work. For consistency, put your final expression in the following form:

$$\boxed{W\left(z^*\right) = A\sin\left(q_0 z^*\right) + B\sinh\left(qz^*\right) + C\sinh\left(q^* z^*\right)} \tag{3}$$

where the new constants A, B, and C are combinations of the C_1, C_2, ... etc. constants from above along with any factors of i or 2 as necessary.

(b) For the odd mode, apply the boundary conditions and construct a 3x3 matrix equation for the solution of A, B, and C, as we did in class for the even mode. Again, show all your work; the algebra gets a little involved, but it is very similar to what we did in class.

(c) Plot the marginal stability curve (thumb curve) for this odd mode. Calculate the minimum Rayleigh number ($\mathrm{Ra}_{\mathrm{crit}}$) for stability for the odd mode, along with the corresponding critical wavenumber K_{crit}. Is the even mode indeed more likely to occur than the odd mode? Explain. *Note*: You may use any computer program you wish, but if you are using Matlab, the Matlab solution for the even mode case is posted on our course's MNE website, main page. I suggest you start with this. If you are having trouble converging, try a different initial guess, as the solution is sensitive to the initial guess.

2. (20 pts) Consider the Orr-Sommerfeld equation. In class we discussed only the temporal mode of instability. Using the same method of normal modes, it is also possible to consider the *spatial* mode of instability, which we will do here.

(a) Re-write the O-S equation in ***standard form*** for an o.d.e. (highest-order term as first term on left, the right-hand side = 0, and group all coefficients together for each order derivative of ϕ). Your final equation should be of the form $\boxed{\phi_{yyyy} - \left[\text{coefficients}\right]\phi_{yy} + \left[\text{coefficients}\right]\phi = 0}$.

(b) Consider the same exponential equations for the normal modes as given in class. After invoking Squires theorem we can simplify the u component of the disturbance to $\boxed{u(x,y,z,t) = \hat{u}(y)e^{ikx - i\omega t}}$ where $\omega = kc$ is the *radian frequency* of the disturbance. For *spatial* instability, discuss which variables (k or ω) should be real and which complex or imaginary.

(c) Describe the eigenvalue nature of the problem. What are the eigenvalues and what are the eigenfunctions? Explain which eigenvalue determines the stability of the flow, and how.

(d) We know that in dimensional terms, the units of ω are [radians/s] and $\omega = 2\pi f$ where f is the physical frequency [cylces/s]. The O-S equation we derived in class, however, is nondimensional. Using the fundamental equation for a wave, namely wave speed = frequency·wavelength, show that for spatial instability wave speed c is in general complex. Generate expressions for the real and imaginary components of c.

(e) Search for Tollmien-Schlichting waves on the Internet and attach a picture or two of some nice flow visualization of T-S waves. (Do not use the same ones shown in class.) Write a little about the photo such as what kind of experiment it was and who performed the experiment. Provide your source(s) (reference(s)).

Note: There is another page. →

3. (25 pts) In this problem, you will derive the linearized disturbance equations for continuity and momentum in cylindrical coordinates (r,θ, z), (u_r, u_θ, u_z). Start with the Navier-Stokes equations (from ME 521) for incompressible flow in cylindrical coordinates:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta)+\frac{\partial}{\partial z}(u_z)=0} \quad (1)$$

$$\boxed{\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^{\,2}}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+g_r+\nu\left(\nabla^2 u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)} \quad (2)$$

$$\boxed{\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}=-\frac{1}{\rho\, r}\frac{\partial p}{\partial \theta}+g_\theta+\nu\left(\nabla^2 u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)} \quad (3)$$

$$\boxed{\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+g_z+\nu\left(\nabla^2 u_z\right)} \quad (4)$$

where the Laplacian is given by $\boxed{\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}}$

(**a**) You can skip the θ-momentum equation (3), unless you really want to do it for fun. Following the notation and the step-by-step procedure given in class, generate the basic equations (1b), (2b), and (4b); total equations (1t), (2t), and (4t); disturbance equations (1d), (2d), and (4d); and linearized disturbance equations (1ℓ), (2ℓ), and (4ℓ), showing all your work and labeling each step. Be sure to number all your intermediate equations (1b), (1t)… For consistency, let the basic state be (U_r, U_θ, U_z). For convenience in grading, put boxes around your final linearized disturbance equations (1ℓ), (2ℓ), and (4ℓ). *Hint*: To save time and writing, you can keep the Laplacian *as is* since it is linear – no need to expand it out. Also, don't use the product rule to expand any of the derivatives (e.g., the first term in Eq. 1).

(**b**) Simplify your linearized disturbance equations for the case of **parallel axisymmetric basic flow**, in which $U_z = U_z(r)$, $U_r = U_\theta = 0$, and $\partial/\partial\theta = 0$ everywhere. Also simplify for the case in which the **fluctuations are also axisymmetric**, meaning that $u_\theta = 0$ everywhere, but u_r and u_z are not zero. This time, expand out the Laplacian terms, but again, don't use the product rule to expand any of the derivatives. You should get 2 terms in the continuity equation, 6 terms in the r-momentum equation, and 6 terms in the z-momentum equation. Call your final axisymmetric equations (1a), (2a), and (4a).

4. (15 pts) Suppose the Rayleigh equation is used to examine the stability of the Blasius boundary layer. We use a numerical approach on a computer with a reasonably refined mesh. [We will not actually do this; this problem is concerned only with the boundary conditions.] The appropriate boundary conditions are $\phi(0) = 0$ and $\phi(\infty) = 0$. Unfortunately, one cannot go to infinity on a computer. Suppose the computational domain extends only to $y/\delta = 5$, where ϕ is *not* zero. For positive real k, find an appropriate boundary condition on ϕ at the upper boundary, $y/\delta = 5$. *Note*: Give a B.C. on ϕ itself, not any of its derivatives. *Hint*: As y increases beyond the edge of the boundary layer, U approaches a constant; this should help you calculate U_y and U_{yy} there. *Another Hint*: You will need to solve a simple differential equation for $\phi(y)$.